QA 196.T3 1890
An elementary sci
[redacted]

Science Library
VANDERBILT
UNIVERSITY LIBRARY
NASHVILLE, TENNESSEE

VUL-0267

QA 18753
196
.T3
1890

DATE DUE

Digitized by the Internet Archive
in 2023

https://archive.org/details/bwb_P8-CTQ-469

# AN ELEMENTARY TREATISE

ON

# QUATERNIONS

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

CAMBRIDGE: DEIGHTON, BELL, AND CO.
LEIPZIG: F. A. BROCKHAUS.

# AN ELEMENTARY TREATISE

# ON

# QUATERNIONS

BY

P. G. TAIT, M.A., SEC. R.S.E.,

HONORARY FELLOW OF ST PETER'S COLLEGE, CAMBRIDGE
PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF EDINBURGH

> . . . . . . . τετρακτύν,
> παγὰν ἀενάου φύσεως ῥιζώματ᾽ ἔχουσαν.

*THIRD EDITION, MUCH ENLARGED*

CAMBRIDGE
AT THE UNIVERSITY PRESS
1890

[*All Rights reserved.*]

062 02

Cambridge:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

# PREFACE.

IN the present edition this work has been very greatly enlarged; to the extent, in fact, of more than one-third. Had I not determined to keep the book in moderate compass it might easily have been doubled in size. A good deal of re-arrangement also has been thought advisable, especially with reference to the elementary uses of $q(\ )q^{-1}$, and of $\nabla$. Prominent among the additions is an entire Chapter, on the Analytical Aspect of Quaternions, which I owe to the unsolicited kindness of Prof. Cayley.

As will be seen by the reader of the former Preface (reprinted below) the point of view which I have, from the first, adopted presents Quaternions as *a Calculus uniquely adapted to Euclidian space,* and therefore specially useful in several of the most important branches of Physical Science. After giving the necessary geometrical and other preliminaries, I have endeavoured to develope it entirely from this point of view: and, though one can scarcely avoid meeting with elegant and often valuable novelties to whatever branch of science he applies such a method, my chief contributions are still those contained in the fifth and the two last Chapters. When, twenty years ago, I published my paper on the application of $\nabla$ to *Green's and other Allied Theorems,* I was under the impression that something similar must have been contemplated, perhaps even mentally worked out, by Hamilton as the subject matter of the (unwritten but promised) concluding section of his *Elements.* It now appears from his *Life* (Vol. III. p. 194) that such was not the case, and thus that I was not in any way anticipated in this application (from my point of view by far the most important yet made) of the Calculus. But a bias in such a special direction of course led to an incomplete because one-sided presentation of the subject. Hence the peculiar importance of the contribution from an Analyst like Prof. Cayley.

It is disappointing to find how little progress has recently been made with the development of Quaternions. One cause, which has been specially active in France, is that workers at the subject have been more intent on modifying the notation, or the mode of presentation of the fundamental principles, than on extending the applications of the Calculus. The earliest offender of this class was the late M. Hoüel who, while availing himself of my permission to reproduce, in his *Théorie des Quantités Complexes*, large parts of this volume, made his pages absolutely repulsive by introducing fancied improvements in the notation. I should not now have referred to this matter (about which I had remonstrated with M. Hoüel) but for a remark made by his friend, M. Laisant, which peremptorily calls for an answer. He says:—"M. Tait...trouve que M. Hoüel a altéré l'œuvre du maître. Perfectionner n'est pas détruire." This appears to be a parody of the saying attributed to Louis XIV.:—"Pardonner n'est pas oublier":—but M. Laisant should have recollected the more important maxim "Le mieux est l'ennemi du bien." A line of Shakespeare might help him:—

> "...with taper-light
> To seek the beauteous eye of heaven to garnish,
> Is wasteful and ridiculous excess."

Even Prof. Willard Gibbs must be ranked as one of the retarders of Quaternion progress, in virtue of his pamphlet on *Vector Analysis*; a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann.

Àpropos of Grassmann, I may advert for a moment to some comparatively recent German statements as to his anticipations &c. of Quaternions. I have given in the last edition of the *Encyc. Brit.* (Art. QUATERNIONS, to which I refer the reader) all that is necessary to shew the absolute baselessness of these statements. The essential points are as follows. Hamilton not only published his theory complete, the year before the first (and extremely imperfect) sketch of the *Ausdehnungslehre* appeared; but had given ten years before, in his protracted study of Sets, the very processes of external and internal multiplication (corresponding to the Vector and Scalar parts of a product of two vectors) which have been put forward as specially the property of Grassmann. The scrupulous care with which Hamilton drew up his account of the work of previous writers (*Lectures*, Preface) is minutely detailed in his correspondence with De Morgan (Hamilton's *Life*, Vol. III.).

Another cause of the slow head-way recently made by Quaternions is undoubtedly to be ascribed to failure in catching the "spirit" of the method:—especially as regards the utter absence of artifice, and the perfect naturalness of every step. To try to patch up a quaternion investigation by having recourse to quasi-Cartesian processes is fatal to progress. A quaternion student loses his self-respect, so to speak, when he thus violates the principles of his Order. Tannhäuser has his representatives in Science as well as in Chivalry! One most insidious and dangerous form of temptation to this dabbling in the unclean thing is pointed out in § 500 below. All who work at the subject should keep before them Hamilton's warning words (*Lectures*, § 513):—

"I regard it as an inelegance and imperfection in this calculus, or rather in the state to which it has hitherto [1853] been unfolded, whenever it becomes, or *seems* to become, necessary to have recourse......to the resources of ordinary algebra, for the SOLUTION OF EQUATIONS IN QUATERNIONS."

As soon as my occupation with teaching and with experimental work perforce ceases to engross the greater part of my time, I hope to attempt, at least, the full quaternion development of several of the theories briefly sketched in the last chapter of this book; provided, of course, that no one have done it in the meantime. From occasional glimpses, hitherto undeveloped, I feel myself warranted in asserting that, immense as are the simplifications introduced by the use of quaternions in the elementary parts of such subjects as Hydrokinetics and Electrodynamics, they are absolutely as nothing compared with those which are to be effected in the higher and (from the ordinary point of view) vastly more complex branches of these fascinating subjects. Complexity is no feature of quaternions themselves, and in presence of their attack (when properly directed) it vanishes from the subject also:—provided, of course, that what we now call complexity depends only upon those space-relations (really simple if rightly approached) which we are in the habit of *making* all but incomprehensible, by surrounding them with our elaborate scaffolding of non-natural cöordinates.

Mr Wilkinson has again kindly assisted me in the revision of the proofs; and they have also been read and annotated by Dr Plarr, the able French Translator of the second edition. Thanks to their valuable aid, I may confidently predict that the present edition will be found comparatively accurate.

With regard to the future of Quaternions, I will merely quote a few words of a letter I received long ago from Hamilton:—

"*Could* anything be simpler or more satisfactory? Don't you *feel*, as well as think, that we are on a *right track*, and shall be *thanked* hereafter? Never mind when."

The special form of thanks which would have been most grateful to a man like Hamilton is to be shewn by practical developments of his magnificent Idea. The award of *this* form of thanks will, I hope, not be long delayed.

P. G. TAIT.

## ADDITIONS, CHANGES, ETC. IN THE PRESENT EDITION.

(*Only the more important are noticed, and they are indicated by the sectional numbers.*)

| | |
|---|---|
| Chap. I. | 31 (k), (m), 40, 43. |
| II. | 51, 89. |
| III. | 105, 108, 116, 119—122, 133—4. |
| IV. | 140 (8)—(12), 145—149. |
| V. | 174, 187, 193—4, 196, 199. |
| VI. | The whole. |
| VIII. | 247, 250, 250*. |
| IX. | 285, 286, 287. |
| X. | 326, 336. |
| XI. | 357—8, 382, 384—6. |
| XII. | 390, 393—403, 407, 458, 473 (*a*)—(*l*), 480, 489, 493, 499, 500, 503, 508—511, 512—13. |

There are large additions to the number of Examples, some in fact to nearly every Chapter. Several of these are of considerable importance; as they contain, or suggest, processes and results not given in the text.

# PREFACE TO THE SECOND EDITION.

To the first edition of this work, published in 1867, the following was prefixed:—

'THE present work was commenced in 1859, while I was a Professor of Mathematics, and far more ready at Quaternion analysis than I can now pretend to be. Had it been then completed I should have had means of testing its teaching capabilities, and of improving it, before publication, where found deficient in that respect.

'The duties of another Chair, and Sir W. Hamilton's wish that my volume should not appear till after the publication of his *Elements*, interrupted my already extensive preparations. I had worked out nearly all the examples of Analytical Geometry in Todhunter's Collection, and I had made various physical applications of the Calculus, especially to Crystallography, to Geometrical Optics, and to the Induction of Currents, in addition to those on Kinematics, Electrodynamics, Fresnel's Wave Surface, &c., which are reprinted in the present work from the *Quarterly Mathematical Journal* and the *Proceedings of the Royal Society of Edinburgh*.

'Sir W. Hamilton, when I saw him but a few days before his death, urged me to prepare my work as soon as possible, his being almost ready for publication. He then expressed, more strongly perhaps than he had ever done before, his profound conviction of the importance of Quaternions to the progress of physical science; and his desire that a really elementary treatise on the subject should soon be published.

'I regret that I have so imperfectly fulfilled this last request of my revered friend. When it was made I was already engaged, along with Sir W. Thomson, in the laborious work of preparing a large Treatise on Natural Philosophy. The present volume has thus been written under very disadvantageous circumstances, especially as I have not found time to work up the mass of materials which I had originally collected for it, but which I had not put into a fit state for publication. I hope, however, that I have to some extent succeeded in producing a thoroughly elementary work, intelligible to any ordinary student; and that the numerous examples I have given, though not specially chosen so as to display the full merits of Quaternions, will yet sufficiently shew their admirable simplicity and naturalness to induce the reader to attack the *Lectures* and the *Elements;* where he will find, in profusion, stores of valuable results, and of elegant yet powerful analytical investigations, such as are contained in the writings of but a very few of the greatest mathematicians. For a succinct account of the steps by which Hamilton was led to the invention of Quaternions, and for other interesting information regarding that remarkable genius, I may refer to a slight sketch of his life and works in the *North British Review* for September 1866.

'It will be found that I have not servilely followed even so great a master, although dealing with a subject which is entirely his own. I cannot, of course, tell in every case what I have gathered from his published papers, or from his voluminous correspondence, and what I may have made out for myself. Some theorems and processes which I have given, though wholly my own, in the sense of having been made out for myself before the publication of the *Elements*, I have since found there. Others also may be, for I have not yet read that tremendous volume completely, since much of it bears on developments unconnected with Physics. But I have endeavoured throughout to point out to the reader all the more important parts of the work which I know to be wholly due to Hamilton. A great part, indeed, may be said to be obvious to any one who has mastered the preliminaries; still I think that, in the two last Chapters especially, a good deal of original matter will be found.

'The volume is essentially a *working* one, and, particularly in the later Chapters, is rather a collection of examples than a detailed treatise on a mathematical method. I have constantly aimed at avoiding too great extension; and in pursuance of this object have omitted many valuable elementary portions of the subject. One of these, the treatment of Quaternion logarithms and exponentials, I greatly regret not having given. But if I had printed all that seemed to me of use or interest to the student, I might easily have rivalled the bulk of one of Hamilton's volumes. The beginner is recommended merely to *read* the first five Chapters, then to *work* at Chapters VI, VII, VIII* (to which numerous easy Examples are appended). After this he may work at the first five, with their (more difficult) Examples; and the remainder of the book should then present no difficulty.

'Keeping always in view, as the great end of every mathematical method, the physical applications, I have endeavoured to treat the subject as much as possible from a geometrical instead of an analytical point of view. Of course, if we premise the properties of $i$, $j$, $k$ merely, it is possible to construct from them the whole system†; just as we deal with the imaginary of Algebra, or, to take a closer analogy, just as Hamilton himself dealt with Couples, Triads, and Sets. This may be interesting to the pure analyst, but it is repulsive to the physical student, who should be led to look upon $i$, $j$, $k$, from the very first as geometric realities, not as algebraic imaginaries.

'The most striking peculiarity of the Calculus is that *multiplication is not generally commutative*, i.e. that $qr$ is in general different from $rq$, $r$ and $q$ being quaternions. Still it is to be remarked that something similar is true, in the ordinary cöordinate methods, of operators and functions: and therefore the student is not wholly unprepared to meet it. No one is puzzled by the fact that $\log . \cos . x$ is not equal to $\cos . \log . x$, or that $\sqrt{\frac{dy}{dx}}$ is not equal to $\frac{d}{dx}\sqrt{y}$. Sometimes, indeed, this rule is most absurdly violated, for it is usual to take $\cos^2 x$ as equal to $(\cos x)^2$, while $\cos^{-1} x$ is not equal to $(\cos x)^{-1}$. No such incongruities appear in Quaternions; but what is true of operators and functions in other methods, that they are not generally commutative, is in Quaternions true in the multiplication of (vector) cöordinates.

---

* [In this edition these Chapters are numbered VII, VIII, IX, respectively—Aug. 1889.]

† This has been done by Hamilton himself, as one among many methods he has employed; and it is also the foundation of a memoir by M. Allégret, entitled *Essai sur le Calcul des Quaternions* (Paris, 1862).

'It will be observed by those who are acquainted with the Calculus that I have, in many cases, not given the shortest or simplest proof of an important proposition. This has been done with the view of including, in moderate compass, as great a variety of methods as possible. With the same object I have endeavoured to supply, by means of the Examples appended to each Chapter, hints (which will not be lost to the intelligent student) of farther developments of the Calculus. Many of these are due to Hamilton, who, in spite of his great originality, was one of the most excellent examiners any University can boast of.

'It must always be remembered that Cartesian methods are mere particular cases of Quaternions, where most of the distinctive features have disappeared; and that when, in the treatment of any particular question, scalars have to be adopted, the Quaternion solution becomes identical with the Cartesian one. Nothing therefore is ever lost, though much is generally gained, by employing Quaternions in preference to ordinary methods. In fact, even when Quaternions degrade to scalars, they give the solution of the most general statement of the problem they are applied to, quite independent of any limitations as to choice of particular cöordinate axes.

'There is one very desirable object which such a work as this may possibly fulfil. The University of Cambridge, while seeking to supply a real want (the deficiency of subjects of examination for mathematical honours, and the consequent frequent introduction of the wildest extravagance in the shape of data for "Problems"), is in danger of making too much of such elegant trifles as Trilinear Coördinates, while gigantic systems like Invariants (which, by the way, are as easily introduced into Quaternions as into Cartesian methods) are quite beyond the amount of mathematics which even the best students can master in three years' reading. One grand step to the supply of this want is, of course, the introduction into the scheme of examination of such branches of mathematical physics as the Theories of Heat and Electricity. But it appears to me that the study of a mathematical method like Quaternions, which, while of immense power and comprehensiveness, is of extraordinary simplicity, and yet requires constant thought in its applications, would also be of great benefit. With it there can be no "shut your eyes, and write down your equations," for mere mechanical dexterity of analysis is certain to lead at once to error on account of the novelty of the processes employed.

'The Table of Contents has been drawn up so as to give the student a short and simple summary of the chief fundamental formulae of the Calculus itself, and is therefore confined to an analysis of the first five [and the two last] chapters.

'In conclusion, I have only to say that I shall be much obliged to any one, student or teacher, who will point out portions of the work where a difficulty has been found; along with any inaccuracies which may be detected. As I have had no assistance in the revision of the proof-sheets, and have composed the work at irregular intervals, and while otherwise laboriously occupied, I fear it may contain many slips and even errors. Should it reach another edition there is no doubt that it will be improved in many important particulars.'

To this I have now to add that I have been equally surprised and delighted by so speedy a demand for a second edition—and the more especially as I have had many pleasing proofs that the work has had considerable circulation in America. There seems now at last to be a reasonable hope that Hamilton's grand invention will soon find its way into the working world of science, to which it is certain to render enormous services, and not be laid aside to be unearthed some centuries hence by some grubbing antiquary.

It can hardly be expected that one whose time is mainly engrossed by physical science, should devote much attention to the purely analytical and geometrical applications of a subject like this; and I am conscious that in many parts of the earlier chapters I have not fully exhibited the simplicity of Quaternions. I hope, however, that the corrections and extensions now made, especially in the later chapters, will render the work more useful for my chief object, the Physical Applications of Quaternions, than it could have been in its first crude form.

I have to thank various correspondents, some anonymous, for suggestions as well as for the detection of misprints and slips of the pen. The only absolute error which has been pointed out to me is a comparatively slight one which had escaped my own notice: a very grave blunder, which I have now corrected, seems not to have been detected by any of my correspondents, so that I cannot be quite confident that others may not exist.

I regret that I have not been able to spare time enough to rewrite the work; and that, in consequence of this, and of the large additions which have been made (especially to the later chapters), the whole will now present even a more miscellaneously jumbled appearance than at first.

It is well to remember, however, that it is quite possible to make a book *too easy* reading, in the sense that the student may read it through several times without feeling those difficulties which (except perhaps in the case of some rare genius) must attend the acquisition of really useful knowledge. It is better to have a rough climb (even cutting one's own steps here and there) than to ascend the dreary monotony of a marble staircase or a well-made ladder. Royal roads to knowledge reach only the particular locality aimed at—and there are no views by the way. It is not on them that pioneers are trained for the exploration of unknown regions.

But I am happy to say that the possible repulsiveness of my early chapters cannot long be advanced as a reason for not attacking this fascinating subject. A still more elementary work than the present will soon appear, mainly from the pen of my colleague Professor KELLAND. In it I give an investigation of the properties of the linear and vector function, based directly upon the Kinematics of Homogeneous Strain, and therefore so different in method from that employed in this work that it may prove of interest to even the advanced student.

Since the appearance of the first edition I have managed (at least partially) to effect the application of Quaternions to line, surface, and volume integrals, such as occur in Hydrokinetics, Electricity, and Potentials generally. I was first attracted to the study of Quaternions by their promise of usefulness in such applications, and, though I have not yet advanced far in this new track, I have got far enough to see that it is certain in time to be of incalculable value to physical science. I have given towards the end of the work all that is necessary to put the student on this track, which will, I hope, soon be followed to some purpose.

One remark more is necessary. I have employed, as the positive direction of rotation, that of the earth about its axis, or about the sun, as seen in our northern latitudes, i.e. that *opposite* to the direction of motion of the hands of a watch. In Sir W. Hamilton's great works the opposite is employed. The student will find no difficulty in passing from the one to the other; but, without previous warning, he is liable to be much perplexed.

With regard to notation, I have retained as nearly as possible that of Hamilton, and where new notation was necessary I have tried to make it as simple and as little incongruous with Hamilton's as possible. This is a part of the work in which great care is absolutely necessary; for, as the subject gains development, fresh notation is inevitably required; and our object must be to make each step such as to defer as long as possible the revolution which must ultimately come.

Many abbreviations are possible, and sometimes very useful in private work; but, as a rule, they are unsuited for print. Every analyst, like every short-hand writer, has his own special contractions; but, when he comes to publish his results, he ought invariably to put such devices aside. If all did not use a common mode of public expression, but each were to print as he is in the habit of writing for his own use, the confusion would be utterly intolerable.

Finally, I must express my great obligations to my friend M. M. U. Wilkinson of Trinity College, Cambridge, for the care with which he has read my proofs, and for many valuable suggestions.

P. G. TAIT.

College, Edinburgh,
*October* 1873.

# CONTENTS.

CHAPTER I.—VECTORS AND THEIR COMPOSITION . pp. 1—28

Sketch of the attempts made to represent geometrically the imaginary of algebra. §§ 1—13.

De Moivre's Theorem interpreted in plane rotation. §§ 7, 8.

Curious speculation of Servois. § 11.

Elementary geometrical ideas connected with relative position. § 14.

Definition of a VECTOR. It may be employed to denote *translation*. Definition of currency. § 16.

Expression of a vector by one symbol, containing implicitly three distinct numbers. Extension of the signification of the symbol $=$. § 18.

The sign $+$ defined in accordance with the interpretation of a vector as representing translation. § 19.

Definition of $-$. It simply reverses the currency of a vector. § 20.

Triangles and polygons of vectors, analogous to those of forces and of simultaneous velocities. § 21.

When two vectors are *parallel* we have

$$\alpha = x\beta. \quad § 22.$$

Any vector whatever may be expressed in terms of three distinct vectors, which are not coplanar, by the formula

$$\rho = x\alpha + y\beta + z\gamma,$$

which exhibits the three numbers on which the vector depends. § 23.

Any vector in the same plane with $\alpha$ and $\beta$ may be written

$$\rho = x\alpha + y\beta. \quad § 24.$$

The equation $\varpi = \rho$, between two vectors, is equivalent to three distinct equations among numbers. § 25.

The *Commutative* and *Associative* Laws hold in the combination of vectors by the signs $+$ and $-$. § 27.

The equation $\rho = x\beta$, where $\rho$ is a variable, and $\beta$ a fixed, vector, represents a line drawn through the origin parallel to $\beta$.

$$\rho = \alpha + x\beta$$

is the equation of a line drawn through the extremity of $\alpha$ and parallel to $\beta$. § 28.

$$\rho = y\alpha + x\beta$$

represents the plane through the origin parallel to $\alpha$ and $\beta$, while

$$\rho = \gamma + y\alpha + x\beta$$

denotes a parallel plane through the point $\gamma$. § 29.

The condition that $\rho$, $\alpha$, $\beta$ may terminate in the same line is

$$p\rho + q\alpha + r\beta = 0,$$

subject to the identical relation

$$p + q + r = 0.$$

Similarly $p\rho + q\alpha + r\beta + s\gamma = 0$,

with $p + q + r + s = 0$,

is the condition that the extremities of four vectors lie in one plane. § 30.

Examples with solutions. Conditions that a vector equation may represent a line, or a surface.

The equation $\rho = \phi t$

represents a curve in space; while

$$\rho = u\phi t$$

is a cone, and $\rho = \phi t + u\alpha$

is a cylinder, both passing through the curve. § 31.

Differentiation of a vector, when given as a function of one number. §§ 32—37.

If the equation of a curve be

$$\rho = \phi(s)$$

where $s$ is the length of the arc, $d\rho$ is a vector-tangent to the curve, and its length is $ds$. §§ 38, 39.

Examples with solutions. §§ 40—43.

Examples to Chapter I. . . . . . . 28—30

## Chapter II.—Products and Quotients of Vectors . 31—57

Here we begin to see what a quaternion is. When two vectors are parallel their quotient is a number. §§ 45, 46.

When they are perpendicular to one another, their quotient is a vector perpendicular to their plane. §§ 47, 64, 72.

When they are neither parallel nor perpendicular the quotient in general involves *four* distinct numbers—and is thus a Quaternion. § 47.

A quaternion $q$ regarded as the operator which turns one vector into another. It is thus decomposable into two factors, whose order is indifferent, the *stretching* factor or Tensor, and the *turning* factor or Versor. These are denoted by $Tq$, and $Uq$. § 48.

The equation $$\beta = q\alpha$$

gives $\frac{\beta}{\alpha} = q$, or $\beta\alpha^{-1} = q$, but *not in general*

$$\alpha^{-1}\beta = q. \quad \S\ 49.$$

$q$ or $\beta\alpha^{-1}$ depends only on the *relative* lengths, and directions, of $\beta$ and $\alpha$. § 50.

*Reciprocal* of a quaternion defined,

$$q = \frac{\beta}{\alpha} \text{ gives } \frac{1}{q} \text{ or } q^{-1} = \frac{\alpha}{\beta},$$

$$T . q^{-1} = \frac{1}{Tq}, \qquad U . q^{-1} = (Uq)^{-1}. \quad \S\ 51.$$

Definition of the *Conjugate* of a quaternion,

$$Kq = (Tq)^2 q^{-1}.$$

and $$qKq = Kq . q = (Tq)^2. \quad \S\ 52.$$

Representation of versors by arcs on the unit-sphere. § 53.

Versor multiplication illustrated by the composition of arcs. The process proved to be *not* generally commutative. § 54.

Proof that $K(qr) = Kr . Kq$. § 55.

Proof of the *Associative* Law of Multiplication

$$p . qr = pq . r. \quad \S\S\ 57—60.$$

[Digression on *Spherical Conics*. § 59*.]

Quaternion addition and subtraction are *commutative*. § 61.

Quaternion multiplication and division are *distributive*. § 62.

Integral powers of a versor are subject to the *Index* Law. § 63.

Composition of *quadrantal* versors in planes at right angles to each other. Calling them $i, j, k$, we have

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$

$$ijk = -1. \quad \S\S\ 64—71.$$

A unit-vector, when employed as a factor, may be considered as a quadrantal versor whose plane is perpendicular to the vector. Hence the equations just written are true of any set of rectangular unit-*vectors* $i, j, k$. § 72.

The product, and also the quotient, of two vectors at right angles to each other is a third perpendicular to both. Hence

$$K\alpha = -\alpha,$$

and $$(T\alpha)^2 = \alpha K\alpha = -\alpha^2. \quad \S\ 73.$$

Every versor may be expressed as a power of some unit-vector. § 74.

Every quaternion may be expressed as a power of a vector. § 75.

The *Index* Law is true of quaternion multiplication and division. § 76.

Quaternion considered as the sum of a Scalar and a Vector.

$$q = \frac{\beta}{\alpha} = x + \gamma = Sq + Vq. \quad \S\S\ 77, 78.$$

Proof that $SKq = Sq$, $VKq = -Vq$, $\Sigma Kq = K\Sigma q$. § 79.

Quadrinomial expression for a quaternion

$$q = w + ix + jy + kz.$$

An equation between quaternions is equivalent to *four* equations between numbers (or scalars). § 80.

Second proof of the *distributive* law of multiplication. §§ 81—83.

Algebraic determination of the constituents of the product and quotient of two vectors. §§ 83, 84.

Second proof of the *associative* law of multiplication. § 85.

Proof of the formulae

$$\begin{aligned} S\alpha\beta &= S\beta\alpha, \\ V\alpha\beta &= -V\beta\alpha, \\ \alpha\beta &= K\beta\alpha, \\ 2S\alpha\beta &= \alpha\beta+\beta\alpha, \\ 2V\alpha\beta &= \alpha\beta-\beta\alpha, \\ S.qr &= S.rq, \\ S.qrs &= S.rsq = S.sqr, \\ S.\alpha\beta\gamma &= S.\beta\gamma\alpha = S.\gamma\alpha\beta = -S.\alpha\gamma\beta = \&c., \end{aligned}$$

$$\left.\begin{array}{r} 2S.\alpha\beta\ldots\phi\chi \\ 2V.\alpha\beta\ldots\phi\chi \end{array}\right\} = \alpha\beta\ldots\phi\chi \pm \chi\phi\ldots\beta\alpha,$$

the upper sign belonging to the scalar if the number of factors is even. §§ 86—89.

Proof of the formulae

$$\begin{aligned} V.\alpha V\beta\gamma &= \gamma S\alpha\beta - \beta S\gamma\alpha, \\ V.\alpha\beta\gamma &= \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta, \\ V.\alpha\beta\gamma &= V.\gamma\beta\alpha, \\ V.V\alpha\beta V\gamma\delta &= \alpha S.\beta\gamma\delta - \beta S.\alpha\gamma\delta, \\ &= \delta S.\alpha\beta\gamma - \gamma S.\alpha\beta\delta, \\ \delta S.\alpha\beta\gamma &= \alpha S.\beta\gamma\delta + \beta S.\gamma\alpha\delta + \gamma S.\alpha\beta\delta, \\ &= V\alpha\beta S\gamma\delta + V\beta\gamma S\alpha\delta + V\gamma\alpha S\beta\delta. \end{aligned}$$

§§ 90—92.

Hamilton's proof that the product of two parallel vectors must be a scalar, and that of perpendicular vectors, a vector; if quaternions are to deal with space indifferently in all directions. § 93.

EXAMPLES TO CHAPTER II. . . . . . . 57—58

CHAPTER III.—INTERPRETATIONS AND TRANSFORMATIONS OF QUATERNION EXPRESSIONS . . . 59—88

If $\theta$ be the angle between two vectors, $\alpha$ and $\beta$, we have

$$S\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}\cos\theta, \qquad S\alpha\beta = -T\alpha T\beta\cos\theta,$$

$$TV\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}\sin\theta, \qquad TV\alpha\beta = \ T\alpha T\beta\sin\theta.$$

§§ 94—96.

Applications to plane trigonometry. § 97.

The condition $S\alpha\beta=0$

shews that $\alpha$ is perpendicular to $\beta$, while

$$V\alpha\beta=0,$$

shews that $\alpha$ and $\beta$ are parallel.

The expression $S.\alpha\beta\gamma$

is the volume of the parallelepiped three of whose conterminous edges are $\alpha, \beta, \gamma$. Hence

$$S.\alpha\beta\gamma=0$$

shews that $\alpha, \beta, \gamma$ are coplanar.

Expression of $S.\alpha\beta\gamma$ as a determinant. §§ 98—102.

Proof that $(Tq)^2 = (Sq)^2 + (TVq)^2$,

and $T(qr) = TqTr$. § 103.

Simple propositions in plane trigonometry. § 104.

Proof that $-\alpha\beta\alpha^{-1}$ is the vector reflected ray, when $\beta$ is the incident ray and $\alpha$ normal to the reflecting surface. § 105.

Interpretation of $\alpha\beta\gamma$ when it is a vector. § 106.

Examples of variety in simple transformations. § 107.

The relation among the distances, two and two, of five points in space. § 108.

De Moivre's Theorem, and Plane Trigonometry. §§ 109—111.

Introduction to spherical trigonometry. §§ 112—116.

Representation, graphic, and by quaternions, of the spherical excess. §§ 117, 118.

Interpretation of the Operator

$$q(\quad)q^{-1}$$

in connection with rotation. Astronomical examples. §§ 119—122.

Loci represented by different equations—points, lines, surfaces, and volumes. §§ 123—126.

Proof that $r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1} = U(rq + KrKq)$. § 127.

Proof of the transformation

$$(S\alpha\rho)^2 + (S\beta\rho)^2 + (S\gamma\rho)^2 = \left(\frac{T(\iota\rho + \rho\kappa)}{\kappa^2 - \iota^2}\right)^2,$$

where $2\begin{Bmatrix}\iota\\ \kappa\end{Bmatrix} = \frac{T\alpha \pm T\gamma}{T\alpha T\gamma}\left(\sqrt{\frac{\beta^2 - \alpha^2}{\gamma^2 - \alpha^2}}\,U\alpha + \sqrt{\frac{\gamma^2 - \beta^2}{\gamma^2 - \alpha^2}}\,U\gamma\right)$. §§ 128, 129.

BIQUATERNIONS. §§ 130—132.

Convenient abbreviations of notation. §§ 133, 134.

EXAMPLES TO CHAPTER III. . . . . . . . . 89—93

CHAPTER IV.—DIFFERENTIATION OF QUATERNIONS . 94—105

Definition of a differential,

$$dr = dFq = \mathcal{L}_{\infty} n \left\{F\left(q + \frac{dq}{n}\right) - Fq\right\},$$

where $dq$ is any quaternion whatever.

We may write $dFq = f(q, dq)$,

where $f$ is linear and homogeneous in $dq$; but we cannot generally write

$$dFq = \mathrm{f}(q)\,dq. \quad §§ 135—138.$$

Definition of the differential of a function of more quaternions than one.

$d(qr) = qdr + dq.r$, but not generally $d(qr) = qdr + rdq$. § 139.

Proofs of fundamental differential expressions:—

$$\frac{dT\rho}{T\rho} = S\frac{d\rho}{\rho},$$

$$\frac{dU\rho}{U\rho} = V\frac{d\rho}{\rho}, \text{ \&c.} \quad § 140.$$

Successive differentiation; Taylor's theorem. §§ 141, 142.

If the equation of a surface be

$$F(\rho) = C,$$

the differential may be written

$$S\nu d\rho = 0,$$

where $\nu$ is a vector normal to the surface. § 144.

Definition of Hamilton's Operator

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

Its effects on simple scalar and vector functions of position. Its square the negative of Laplace's Operator. Expressions for the condensation and rotation due to a displacement. Application to fluxes, and to normals to surfaces. Precautions necessary in its use. §§ 145—149.

EXAMPLES TO CHAPTER IV. . . . . . . 106, 107

CHAPTER V.—THE SOLUTION OF EQUATIONS OF THE FIRST DEGREE . . . . . . . 108—141

The most general equation of the first degree in an unknown quaternion $q$, may be written

$$\Sigma V.aqb + S.cq = d,$$

where $a$, $b$, $c$, $d$ are given quaternions. Elimination of $Sq$, and reduction to the vector equation

$$\phi\rho = \Sigma.\alpha S\beta\rho = \gamma. \quad \text{§§ 150, 151.}$$

General proof that $\phi^3\rho$ is expressible as a linear function of $\rho$, $\phi\rho$, and $\phi^2\rho$. § 152.

Value of $\phi$ for an ellipsoid, employed to illustrate the general theory. §§ 153—155.

Hamilton's solution of $\phi\rho = \gamma$.

If we write $S\sigma\phi\rho = S\rho\phi'\sigma$,

the functions $\phi$ and $\phi'$ are said to be conjugate, and

$$m\phi^{-1}V\lambda\mu = V\phi'\lambda\phi'\mu.$$

Proof that $m$, whose value may be written as

$$\frac{S.\phi'\lambda\phi'\mu\phi'\nu}{S.\lambda\mu\nu},$$

is the same for all values of $\lambda$, $\mu$, $\nu$. §§ 156—158.

Proof that if $m_g = m - m_1 g + m_2 g^2 - g^3$,

where $$m_1 = \frac{S(\lambda\phi'\mu\phi'\nu + \phi'\lambda\mu\phi'\nu + \phi'\lambda\phi'\mu\nu)}{S.\lambda\mu\nu},$$

and $$m_2 = \frac{S(\lambda\mu\phi'\nu + \phi'\lambda\mu\nu + \lambda\phi'\mu\nu)}{S.\lambda\mu\nu},$$

(which, like $m$, are *Invariants*,)

then $$m_g(\phi - g)^{-1}V\lambda\mu = (m\phi^{-1} - g\chi + g^2)V\lambda\mu.$$

Also that $$\chi = m_2 - \phi,$$

whence the final form of solution

$$m\phi^{-1} = m_1 - m_2\phi + \phi^2. \quad \text{§§ 159, 160.}$$

Examples. §§ 161—173.

The fundamental cubic

$$\phi^3 - m_2\phi^2 + m_1\phi - m = (\phi - g_1)(\phi - g_2)(\phi - g_3) = 0.$$

When $\phi$ is its own conjugate, the roots of the cubic are real; and the equation

$$V\rho\phi\rho = 0,$$

or

$$(\phi - g)\,\rho = 0,$$

is satisfied by a set of three real and mutually perpendicular vectors.

Geometrical interpretation of these results. §§ 174—178.

Proof of the transformation of the self-conjugate linear and vector function

$$\phi\rho = f\rho + hV.(i + ek)\,\rho\,(i - ek)$$

where

$$(\phi - g_1)\,i = 0,$$

$$(\phi - g_3)\,k = 0,$$

$$e^2 = \frac{g_2 - g_3}{g_1 - g_2},$$

$$f = \tfrac{1}{2}(g_1 + g_3),$$

$$h = -\tfrac{1}{2}(g_1 - g_2).$$

Another transformation is

$$\phi\rho = a\alpha V\alpha\rho + b\beta S\beta\rho. \quad \text{§§ 179—181.}$$

Other properties of $\phi$. Proof that

$$S\rho\,(\phi - g)^{-1}\rho = 0, \quad \text{and} \quad S\rho\,(\phi - h)^{-1}\rho = 0$$

represent the same surface if

$$mS\rho\phi^{-1}\rho - gh\rho^2.$$

Proof that when $\phi$ is not self-conjugate

$$\dot{\phi}\rho = \phi'\rho + V\epsilon\rho. \quad \text{§§ 182—184.}$$

Proof that, if $q = \alpha\phi\alpha + \beta\phi\beta + \gamma\phi\gamma$,

where $\alpha$, $\beta$, $\gamma$ are any rectangular unit-vectors whatever, we have

$$Sq = -m_2, \qquad Vq = \epsilon,$$

where

$$V\epsilon\rho = \tfrac{1}{2}(\phi - \phi')\,\rho.$$

This quaternion can be expressed in the important form

$$q = \nabla\phi\rho. \quad \text{§ 185.}$$

A non-conjugate linear and vector function of a vector differs from a self-conjugate one solely by a term of the form

$$V\epsilon\rho. \quad \text{§ 186.}$$

Graphic determination of the conditions that there may be three real vector solutions of

$$V\rho\phi\rho = 0. \quad \text{§ 187.}$$

Degrees of indeterminateness of the solution of a quaternion equation—Examples. §§ 188—191.

The linear function of a quaternion is given by a symbolical biquadratic. § 192.

Particular forms of linear equations. Differential of the *n*th root of a quaternion. §§ 193—196.

A quaternion equation of the *m*th degree in general involves a scalar equation of degree $m^4$. § 197.

Solution of the equation $q^2 = qa + b$. § 198.

EXAMPLES TO CHAPTER V. . . . . . . . . 142—145

CHAPTER VI.—SKETCH OF THE ANALYTICAL THEORY OF QUATERNIONS . . . . . . . . 146—159

CHAPTER VII.—GEOMETRY OF THE STRAIGHT LINE AND PLANE . . . . . . . . 160—174

EXAMPLES TO CHAPTER VII. . . . . . . . . 175—177

CHAPTER VIII.—THE SPHERE AND CYCLIC CONE . 178—198

EXAMPLES TO CHAPTER VIII. . . . . . . 199—201

CHAPTER IX.—SURFACES OF THE SECOND DEGREE . 202—224

EXAMPLES TO CHAPTER IX. . . . . . . . 224—229

CHAPTER X.—GEOMETRY OF CURVES AND SURFACES 230—269

EXAMPLES TO CHAPTER X. . . . . . . . . 270—278

CHAPTER XI.—KINEMATICS . . . . . . 279—304

### A. *Kinematics of a Point.* §§ 354—366.

If $\rho=\phi t$ be the vector of a moving point in terms of the time, $\dot{\rho}$ is the vector velocity, and $\ddot{\rho}$ the vector acceleration.

$\sigma=\dot{\rho}=\phi'(t)$ is the equation of the Hodograph.

$\ddot{\rho}=\dot{v}\rho'+v^2\rho''$ gives the normal and tangential accelerations.

$V\rho\rho=0$ if acceleration directed to a point, whence $V\rho\rho=\gamma$.

Examples.—Planetary acceleration. Here the equation is

$$\ddot{\rho}=\frac{\mu U\rho}{T\rho^2},$$

giving $V\rho\dot{\rho}=\gamma$; whence the hodograph is

$$\dot{\rho}=\epsilon\gamma^{-1}-\mu U\rho\,.\,\gamma^{-1},$$

and the orbit is the section of

$$\mu T\rho=S\epsilon\,(\gamma^2\epsilon^{-1}-\rho)$$

by the plane $$S\gamma\rho=0.$$

Cotes' Spirals, Epitrochoids, &c. §§ 354—366.

### B. *Kinematics of a Rigid System.* §§ 367—375.

Rotation of a rigid system. Composition of rotations. If the position of a system at time $t$ is derived from the initial position by $q(\quad)q^{-1}$, the instantaneous axis is

$$\epsilon=2V\dot{q}q^{-1}.$$

Rodrigues' cöordinates. §§ 367—375.

### *C. Kinematics of a Deformable System.* §§ 376—386.

Homogeneous strain. Criterion of pure strain. Separation of the rotational from the pure part. Extraction of the square root of a strain. A strain $\phi$ is equivalent to a pure strain $\sqrt{\phi'\phi}$ followed by the rotation $\phi/\sqrt{\phi'\phi}$. Simple Shear. §§ 376—383.

Displacements of systems of points. Consequent condensation and rotation. Preliminary about the use of $\nabla$ in physical questions. For displacement $\sigma$, the strain function is

$$\phi\tau = \tau - S\tau\nabla \, . \, \sigma. \quad §§\ 384—386.$$

### *D. Axes of Inertia.* § 387.

Moment of inertia. Binet's Theorem. § 387.

EXAMPLES TO CHAPTER XI. . . . . . . . 304—308

CHAPTER XII.—PHYSICAL APPLICATIONS . . 309—409

### *A. Statics of a Rigid System.* §§ 389—403.

Condition of equilibrium of a rigid system is $\Sigma S \, . \, \beta\delta\alpha = 0$, where $\beta$ is a vector force, $\alpha$ its point of application. Hence the usual six equations in the form $\Sigma\beta = 0$, $\Sigma V\alpha\beta = 0$. Central axis. Minding's Theorem, &c. §§ 389—403.

### *B. Kinetics of a Rigid System.* §§ 404—425.

For the motion of a rigid system

$$\Sigma S\,(m\ddot{\alpha} - \beta)\,\delta\alpha = 0,$$

whence the usual forms. Theorems of Poinsot and Sylvester. The equation

$$2\dot{q} = q\phi^{-1}(q^{-1}\gamma q),$$

where $\gamma$ is given in terms of $t$ and $q$ if forces act, but is otherwise constant, contains the whole theory of the motion of a rigid body with one point fixed. Reduction to the ordinary form

$$\frac{dt}{2} = \frac{dw}{W} = \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}\,.$$

Here, if no forces act, $W$, $X$, $Y$, $Z$ are *homogeneous* functions of the third degree in $w$, $x$, $y$, $z$. §§ 404—425.

### *C. Special Kinetic Problems.* §§ 426—430.

Precession and Nutation. General equation of motion of simple pendulum. Foucault's pendulum. §§ 426—430.

### *D. Geometrical and Physical Optics.* §§ 431—452.

Problem on reflecting surfaces. § 431.

Fresnel's Theory of Double Refraction. Various forms of the equation of Fresnel's Wave-surface;

$$S\,.\,\rho(\phi - \rho^2)^{-1}\rho = -1, \quad T(\rho^{-2} - \phi^{-1})^{-\frac{1}{2}}\rho = 0, \quad 1 = -p\rho^2 \mp (T \pm S)\,V\lambda\rho V\mu\rho.$$

The conical cusps and circles of contact. Lines of vibration, &c. §§ 432—452.

### *E. Electrodynamics.* §§ 453—472.

Electrodynamics. The vector action of a closed circuit on an element of current $\alpha_1$ is proportional to $V\alpha_1\beta$ where

$$\beta = \int \frac{V\alpha d\alpha}{T\alpha^3} = \int \frac{dU\alpha}{\alpha},$$

the integration extending round the circuit. It can also be expressed as $-\nabla\Omega$, where $\Omega$ is the spherical angle subtended by the circuit. This is a many-valued function. Special case of a circular current. Mutual action of two closed circuits, and of solenoids. Mutual action of magnets. Potential of a closed circuit. Magnetic curves. §§ 453—472.

### *F. General Expressions for the Action between Linear Elements.* § 473.

Assuming Ampère's data I, II, III, what is the most general expression for the mutual action between two elements? Particular cases, determined by a fourth assumption. Solution of the problem when I and II, alone, are assumed. Special cases, including v. Helmholtz' form. § 473 (*a*)...(*l*).

### *G. Application of* $\nabla$ *to certain Physical Analogies.* §§ 474—478.

The effect of a current-element on a magnetic particle is analogous to displacement produced by external forces in an elastic solid, while that of a small *closed* circuit (or magnet) is analogous to the corresponding vector rotation.

### *H. Elementary Properties of* $\nabla$. §§ 479—481.

$$Sd\rho\nabla = -d = Sd\sigma\nabla_\sigma$$

where $\sigma = i\xi + j\eta + k\zeta$ and $\nabla_\sigma = i\frac{d}{d\xi} + j\frac{d}{d\eta} + k\frac{d}{d\zeta}$;

so that, if $$d\sigma = \phi d\rho,$$ $$\nabla_\sigma = \phi'^{-1}\nabla.$$

### *J. Applications of* $\nabla$ *to Line, Surface and Volume Integrals.* §§ 482—501.

Proof of the fundamental theorem for comparing an integral over a closed surface with one through its content

$$\iiint S\nabla\sigma d\varsigma = \iint S\sigma U\nu ds.$$

Hence Green's Theorem. Examples in potentials and in conduction of heat. Limitations and ambiguities. Determination of the displacement in a fluid when the consequent rotation is given. §§ 482—494.

Similar theorem for double and single integrals

$$\int S\sigma d\rho = \iint S \,.\, U\nu\nabla\sigma ds. \quad \text{§§ 495—497.}$$

The fundamental forms, of which all the others are simple consequences, are

$$\int d\rho u = \iint ds V(U\nu\nabla)u,$$

$$\iiint \nabla u d\varsigma = \iint U\nu u ds. \quad \text{§§ 498, 499.}$$

The first is a particular case of the second. § 500.

Expression for a surface in terms of an integral through the enclosed volume. § 499.

### *K. Application of the $\nabla$ Integrals to Magnetic &c. Problems.*
### §§ 502—506.

Volume and surface distributions due to a given magnetic field. Solenoidal and Lamellar distributions. § 502. Magnetic Induction and Vector potential. § 503. Ampère's *Directrice*. § 504. Gravitation potential of homogeneous solid in terms of a surface-integral. § 505.

### *L. Application of $\nabla$ to the Stress-Function.* §§ 507—511.

When there are no molecular couples the stress-function is self-conjugate. § 507. Properties of this function which depend upon the equilibrium of any definite portion of the solid, *as a whole*. § 508. Expression for the stress-function, in terms of displacement, when the solid is isotropic:—

$$\phi\omega = -n(S\omega\nabla . \sigma + \nabla S\omega\sigma) - (c - \tfrac{2}{3}n)\,\omega S\nabla\sigma.$$

Examples. § 509. Work due to displacement in *any* elastic solid. Green's 21 elastic coefficients. § 511.

### *M. The Hydrokinetic Equations.* §§ 512, 513.

Equation of continuity, for displacement $\sigma$:—

$$\frac{\partial e}{\partial t} = \frac{de}{dt} - S\sigma\nabla . e = eS\nabla\sigma.$$

Equation for rate of change of momentum in unit volume:—

$$\frac{\partial\sigma}{\partial t} = -\nabla\left(P + \int\frac{dp}{e}\right) = -\nabla Q. \quad \S\ 512.$$

Term introduced by Viscosity:—

$$\propto\nabla^2\sigma + \tfrac{1}{3}\nabla S\nabla\sigma. \quad (\textit{Miscellaneous Examples}, 36.)$$

Helmholtz's Transformation for Vortex-motion:—

$$\frac{\partial}{\partial t} V\nabla\sigma = V . \nabla V . \sigma V\nabla_1\sigma_1.$$

Thomson's Transformation for Circulation:—

$$-\frac{\partial f}{\partial t} = \Big[Q - \tfrac{1}{2}v^2\Big]_a^\rho, \text{ where } f = -\int_a^\rho S\sigma d\rho.$$

### *N. Use of $\nabla$ in connection with Taylor's Theorem.* §§ 514—517.

Proof that

$$\epsilon^{-S\sigma\nabla} f(\rho) = f(\rho + \sigma).$$

Applications and consequences. Separation of symbols of operation and their treatment as quantities.

### *O. Applications of $\nabla$ in connection with the Calculus of Variations.*
### §§ 518—527.

If $$A = \int QTd\rho$$
we have $$\delta A = -[QSUd\rho\delta\rho] + \int\{\delta QTd\rho + S . \delta\rho(QU\delta\rho)\},$$
whence, if $A$ is a maximum or minimum,

$$\frac{d}{ds}(Q\rho') - \nabla Q = 0.$$

Applications to Varying Action, Brachistochrones, Catenaries, &c. §§ 518—526. Thomson's Theorem that there is one, and but one, solution of the equation

$$S . \nabla(e^2\nabla u) = 4\pi r. \quad \S\ 527.$$

MISCELLANEOUS EXAMPLES . . . . . . . . 409—421.

# QUATERNIONS.

## CHAPTER I.

### VECTORS, AND THEIR COMPOSITION.

1. FOR at least two centuries the geometrical representation of the negative and imaginary algebraic quantities, $-1$ and $\sqrt{-1}$ has been a favourite subject of speculation with mathematicians. The essence of almost all of the proposed processes consists in employing such expressions to indicate the DIRECTION, not the *length*, of lines.

2. Thus it was long ago seen that if positive quantities were measured off in one direction along a fixed line, a useful and lawful convention enabled us to express negative quantities of the same kind by simply laying them off on the same line in the opposite direction. This convention is an essential part of the Cartesian method, and is constantly employed in Analytical Geometry and Applied Mathematics.

3. Wallis, towards the end of the seventeenth century, proposed to represent the impossible roots of a quadratic equation by going *out of* the line on which, if real, they would have been laid off. This construction is equivalent to the consideration of $\sqrt{-1}$ as a directed unit-line perpendicular to that on which real quantities are measured.

4. In the usual notation of Analytical Geometry of two dimensions, when rectangular axes are employed, this amounts to reckoning each unit of length along $Oy$ as $+\sqrt{-1}$, and on $Oy'$ as $-\sqrt{-1}$; while on $Ox$ each unit is $+1$, and on $Ox'$ it is $-1$.

If we look at these four lines in circular order, i.e. in the order of positive rotation (that of the northern hemisphere of the earth about its axis, or *opposite* to that of the hands of a watch), they give

$$1, \quad \sqrt{-1}, \quad -1, \quad -\sqrt{-1}.$$

In this series each expression is derived from that which precedes it by multiplication by the factor $\sqrt{-1}$. Hence we may consider $\sqrt{-1}$ as an operator, analogous to a handle perpendicular to the plane of $xy$, whose effect on any line in that plane is to make it rotate (positively) about the origin through an angle of 90°.

5. In such a system, (which seems to have been first developed, in 1805, by Buée) a point in the plane of reference is defined by a single imaginary expression. Thus $a+b\sqrt{-1}$ may be considered as a single quantity, denoting the point, $P$, whose cöordinates are $a$ and $b$. Or, it may be used as an expression for the line $OP$ joining that point with the origin. In the latter sense, the expression $a+b\sqrt{-1}$ implicitly contains the *direction*, as well as the *length*, of this line; since, as we see at once, the direction is inclined at an angle $\tan^{-1} b/a$ to the axis of $x$, and the length is $\sqrt{a^2+b^2}$. Thus, say we have

$$OP = a + b\sqrt{-1};$$

the line $OP$ considered as that by which we pass from one extremity, $O$, to the other, $P$. In this sense it is called a VECTOR. Considering, in the plane, any other vector,

$$OQ = a' + b'\sqrt{-1};$$

the addition of these two lines obviously gives

$$OR = a + a' + (b + b')\sqrt{-1};$$

and we see that the sum is the diagonal of the parallelogram on $OP$, $OQ$. This is the law of the composition of simultaneous velocities; and it contains, of course, the law of subtraction of one directed line from another.

6. Operating on the first of these symbols by the factor $\sqrt{-1}$, it becomes $-b+a\sqrt{-1}$; and now, of course, denotes the point whose $x$ and $y$ cöordinates are $-b$ and $a$; or the line joining this point with the origin. The length is still $\sqrt{a^2+b^2}$, but the angle the line makes with the axis of $x$ is $\tan^{-1}(-a/b)$; which is evidently greater by $\pi/2$ than before the operation.

7. De Moivre's Theorem tends to lead us still further in the same direction. In fact, it is easy to see that if we use, instead of $\sqrt{-1}$, the more general factor $\cos\alpha + \sqrt{-1}\sin\alpha$, its effect on any line is to turn it through the (positive) angle $\alpha$ in the plane of $x$, $y$. [Of course the former factor, $\sqrt{-1}$, is merely the particular case of this, when $\alpha = \pi/2$.]

Thus $(\cos\alpha + \sqrt{-1}\sin\alpha)(a + b\sqrt{-1})$

$$= a\cos\alpha - b\sin\alpha + \sqrt{-1}\,(a\sin\alpha + b\cos\alpha),$$

by direct multiplication. The reader will at once see that the new form indicates that a rotation through an angle $\alpha$ has taken place, if he compares it with the common formulæ for turning the coördinate axes through a given angle. Or, in a less simple manner, thus—

$$\text{Length} = \sqrt{(a\cos\alpha - b\sin\alpha)^2 + (a\sin\alpha + b\cos\alpha)^2}$$
$$= \sqrt{a^2 + b^2} \qquad \text{as before.}$$

Inclination to axis of $x$

$$= \tan^{-1}\frac{a\sin\alpha + b\cos\alpha}{a\cos\alpha - b\sin\alpha} = \tan^{-1}\frac{\tan\alpha + \frac{b}{a}}{1 - \frac{b}{a}\tan\alpha}$$

$$= \alpha + \tan^{-1} b/a.$$

8. We see now, as it were, *why* it happens that

$$(\cos\alpha + \sqrt{-1}\sin\alpha)^m = \cos m\alpha + \sqrt{-1}\sin m\alpha.$$

In fact, the first operator produces $m$ successive rotations in the same direction, each through the angle $\alpha$; the second, a single rotation through the angle $m\alpha$.

9. It may be interesting, at this stage, to anticipate so far as to remark that in the theory of Quaternions the analogue of

$$\cos\theta + \sqrt{-1}\sin\theta$$

is

$$\cos\theta + \varpi\sin\theta,$$

where

$$\varpi^2 = -1.$$

Here, however, $\varpi$ is not the algebraic $\sqrt{-1}$, but is *any directed unit-line* whatever in space.

10. In the present century Argand, Warren, Mourey, and others, extended the results of Wallis and Buée. They attempted

to express as a line the product of two lines each represented by a symbol such as $a+b\sqrt{-1}$. To a certain extent they succeeded, but all their results remained confined to two dimensions.

The product, $\Pi$, of two such lines was defined as the fourth proportional to unity and the two lines, thus

$$1 : a+b\sqrt{-1} :: a'+b'\sqrt{-1} : \Pi,$$

or

$$\Pi = (aa'-bb') + (a'b+b'a)\sqrt{-1}.$$

The length of $\Pi$ is obviously the product of the lengths of the factor lines; and its direction makes an angle with the axis of $x$ which is the sum of those made by the factor lines. From this result the quotient of two such lines follows immediately.

11. A very curious speculation, due to Servois and published in 1813 in Gergonne's *Annales*, is one of the very few, so far as has been discovered, in which a well-founded guess at a possible mode of extension to three dimensions is contained. Endeavouring to extend to *space* the form $a+b\sqrt{-1}$ for the plane, he is guided by analogy to write for a directed unit-line in space the form

$$p\cos\alpha + q\cos\beta + r\cos\gamma,$$

where $\alpha$, $\beta$, $\gamma$ are its inclinations to the three axes. He perceives easily that $p$, $q$, $r$ must be *non-reals:* but, he asks, "seraient-elles *imaginaires* réductibles à la forme générale $A+B\sqrt{-1}$?" The $i$, $j$, $k$ of the Quaternion Calculus furnish an answer to this question. (See Chap. II.) But it may be remarked that, in applying the idea to lines in a plane, a vector $OP$ will no longer be represented (as in § 5) by

$$OP = a+b\sqrt{-1},$$

but by

$$OP = pa+qb.$$

And if, similarly,

$$OQ = pa'+qb',$$

the addition of these two lines gives for $OR$ (which retains its previous signification)

$$OR = p(a+a') + q(b+b').$$

12. Beyond this, few attempts were made, or at least recorded, in earlier times, to extend the principle to space of three dimensions; and, though many such had been made before 1843, none, with the single exception of Hamilton's, have resulted in simple, practical methods; all, however ingenious, seeming to lead almost at once to processes and results of fearful complexity.

For a lucid, complete, and most impartial statement of the claims of his predecessors in this field we refer to the Preface to Hamilton's *Lectures on Quaternions*. He there shews how his long protracted investigations of *Sets* culminated in this unique system of tridimensional-space geometry.

13. It was reserved for Hamilton to discover the use and properties of a class of symbols which, though all in a certain sense square roots of $-1$, may be considered as *real* unit lines, tied down to no particular direction in space; the expression for a vector is, or may be taken to be,

$$\rho = ix + jy + kz;$$

but such vector is considered in connection with an *extraspatial* magnitude $w$, and we have thus the notion of a QUATERNION

$$w + \rho.$$

This is the fundamental notion in the singularly elegant, and enormously powerful, Calculus of Quaternions.

While the schemes for using the algebraic $\sqrt{-1}$ to indicate direction make one direction in space expressible by real numbers, the remainder being imaginaries of some kind, and thus lead to expressions which are heterogeneous; Hamilton's system makes all directions in space equally imaginary, or rather equally real, thereby ensuring to his Calculus the power of dealing with space indifferently in all directions.

In fact, as we shall see, the Quaternion method is independent of axes or any supposed directions in space, and takes its reference lines solely from the problem it is applied to.

14. But, for the purpose of elementary exposition, it is best to begin by assimilating it as closely as we can to the ordinary Cartesian methods of Geometry of Three Dimensions, with which the student is supposed to be, to some extent at least, acquainted. Such assistance, it will be found, can (as a rule) soon be dispensed with; and Hamilton regarded any apparent necessity for an occasional recurrence to it, in higher applications, as an indication of imperfect development in the proper methods of the new Calculus.

We commence, therefore, with some very elementary geometrical ideas, relating to the theory of vectors in space. It will subsequently appear how we are thus led to the notion of a Quaternion.

15. Suppose we have two points $A$ and $B$ in *space*, and suppose $A$ given, on how many numbers does $B$'s relative position depend?

If we refer to Cartesian cöordinates (rectangular or not) we find that the data required are the excesses of $B$'s three cöordinates over those of $A$. Hence *three* numbers are required.

Or we may take polar cöordinates. To define the moon's position with respect to the earth we must have its Geocentric Latitude and Longitude, *or* its Right Ascension and Declination, and, in addition, its distance or radius-vector. *Three* again.

16. Here it is to be carefully noticed that nothing has been said of the *actual* cöordinates of either $A$ or $B$, or of the earth and moon, in space; it is only the *relative* cöordinates that are contemplated.

Hence any expression, as $\overline{AB}$, denoting a line considered with reference to direction and currency as well as length, (whatever may be its actual position in space) contains implicitly *three* numbers, and all lines parallel and equal to $AB$, and concurrent with it, depend in the same way upon the same three. Hence, *all lines which are equal, parallel, and concurrent, may be represented by a common symbol, and that symbol contains three distinct numbers.* In this sense a line is called a VECTOR, since by it we pass from the one extremity, $A$, to the other, $B$; and it may thus be considered as an instrument which *carries* $A$ to $B$: so that a vector may be employed to indicate a definite *translation* in space.

[The term "currency" has been suggested by Cayley for use instead of the somewhat vague suggestion sometimes taken to be involved in the word "direction." Thus parallel lines have the same direction, though they may have similar or opposite currencies. The definition of a vector essentially includes its currency.]

17. We may here remark, once for all, that in establishing a new Calculus, we are at liberty to give any definitions whatever of our symbols, provided that no two of these interfere with, or contradict, each other, and in doing so in Quaternions *simplicity* and (so to speak) *naturalness* were the inventor's aim.

18. Let $\overline{AB}$ be represented by $\alpha$, we know that $\alpha$ involves *three* separate numbers, and that these depend solely upon the position of $B$ *relatively* to $A$. Now if $CD$ be equal in length to $AB$

and if these lines be parallel, and have the same currency, we may evidently write

$$\overline{CD} = \overline{AB} = \alpha,$$

where it will be seen that the sign of *equality* between vectors contains implicitly *equality in length, parallelism in direction*, and *concurrency*. So far we have *extended* the meaning of an algebraical symbol. And it is to be noticed that an equation between vectors, as

$$\alpha = \beta,$$

contains *three* distinct equations between mere numbers.

19. We must now define + (and the meaning of − will follow) in the new Calculus. Let $A$, $B$, $C$ be any three points, and (with the above meaning of =) let

$$\overline{AB} = \alpha, \quad \overline{BC} = \beta, \quad \overline{AC} = \gamma.$$

If we define + (in accordance with the idea (§ 16) that a vector represents a *translation*) by the equation

$$\alpha + \beta = \gamma,$$

or

$$\overline{AB} + \overline{BC} = \overline{AC},$$

we contradict nothing that precedes, but we at once introduce the idea that *vectors are to be compounded, in direction and magnitude, like simultaneous velocities.* A reason for this may be seen in another way if we remember that by *adding* the (algebraic) differences of the Cartesian cöordinates of $B$ and $A$, to those of the cöordinates of $C$ and $B$, we get those of the cöordinates of $C$ and $A$. Hence these cöordinates enter *linearly* into the expression for a vector. (See, again, § 5.)

20. But we also see that if $C$ and $A$ coincide (and $C$ may be *any* point)

$$\overline{AC} = 0,$$

for no vector is then required to carry $A$ to $C$. Hence the above relation may be written, in this case,

$$\overline{AB} + \overline{BA} = 0,$$

or, introducing, and by the same act defining, the symbol −,

$$\overline{BA} = -\overline{AB}.$$

Hence, *the symbol −, applied to a vector, simply shews that its currency is to be reversed.*

And this is consistent with all that precedes; for instance,

$$\overline{AB}+\overline{BC}=\overline{AC},$$

and

$$\overline{AB}=\overline{AC}-\overline{BC},$$

or

$$=\overline{AC}+\overline{CB},$$

are evidently but different expressions of the same truth.

21. In any triangle, $ABC$, we have, of course,

$$\overline{AB}+\overline{BC}+\overline{CA}=0;$$

and, in any closed polygon, whether plane or gauche,

$$\overline{AB}+\overline{BC}+\ldots\ldots+\overline{YZ}+\overline{ZA}=0.$$

In the case of the polygon we have also

$$\overline{AB}+\overline{BC}+\ldots\ldots+\overline{YZ}=\overline{AZ}.$$

These are the well-known propositions regarding composition of velocities, which, by Newton's second law of motion, give us the geometrical laws of composition of forces acting at one point.

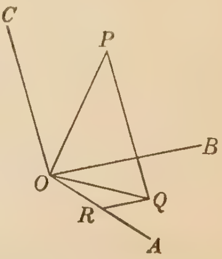
22. If we compound any number of *parallel* vectors, the result is obviously a numerical multiple of any one of them.

Thus, if $A$, $B$, $C$ are in one straight line,

$$\overline{BC}=x\,\overline{AB};$$

where $x$ is a number, positive when $B$ lies between $A$ and $C$, otherwise negative: but such that its numerical value, independent of sign, is the ratio of the length of $BC$ to that of $AB$. This is at once evident if $AB$ and $BC$ be commensurable; and is easily extended to incommensurables by the usual *reductio ad absurdum.*

23. An important, but almost obvious, proposition is that *any vector may be resolved, and in one way only, into three components parallel respectively to any three given vectors, no two of which are parallel, and which are not parallel to one plane.*



Let $OA$, $OB$, $OC$ be the three fixed vectors, $OP$ any other vector. From $P$ draw $PQ$ parallel to $CO$, meeting the plane $BOA$ in $Q$. [There must be a definite point $Q$, else $PQ$, and therefore $CO$, would be parallel to $BOA$, a case specially excepted.] From $Q$ draw $QR$ parallel to $BO$, meeting $OA$ in $R$. Then we have $\overline{OP}=\overline{OR}+\overline{RQ}+\overline{QP}$ (§ 21), and these components are respectively parallel to the three given vectors. By § 22 we may express $\overline{OR}$ as a numerical multiple of $\overline{OA}$, $\overline{RQ}$ of $\overline{OB}$, and $\overline{QP}$ of $\overline{OC}$. Hence we have, generally, for any vector in terms of three fixed non-coplanar vectors, $\alpha$, $\beta$, $\gamma$,

$$\overline{OP} = \rho = x\alpha + y\beta + z\gamma,$$

which exhibits, in one form, the *three* numbers on which a vector depends (§ 16). Here $x$, $y$, $z$ are perfectly definite, and can have but single values.

24. Similarly any vector, as $\overline{OQ}$, in the same plane with $OA$ and $OB$, can be resolved (in one way only) into components $\overline{OR}$, $\overline{RQ}$, parallel respectively to $\overline{OA}$ and $\overline{OB}$; so long, at least, as these two vectors are not parallel to each other.

25. There is particular advantage, in certain cases, in employing a series of three *mutually perpendicular unit-vectors* as lines of reference. This system Hamilton denotes by $i, j, k$.

Any other vector is then expressible as

$$\rho = xi + yj + zk.$$

Since $i$, $j$, $k$ are unit-vectors, $x$, $y$, $z$ are here the lengths of conterminous edges of a rectangular parallelepiped of which $\rho$ is the vector-diagonal; so that the length of $\rho$ is, in this case,

$$\sqrt{x^2 + y^2 + z^2}.$$

Let
$$\varpi = \xi i + \eta j + \zeta k$$

be any other vector, then (by the proposition of § 23) the vector equation
$$\rho = \varpi$$

obviously involves the following three equations among numbers,

$$x = \xi, \qquad y = \eta, \qquad z = \zeta.$$

Suppose $i$ to be drawn eastwards, $j$ northwards, and $k$ upwards, this is equivalent merely to saying that *if two points coincide, they are equally to the east (or west) of any third point, equally to the north (or south) of it, and equally elevated above (or depressed below) its level.*

26. It is to be carefully noticed that it is only when $\alpha, \beta, \gamma$ are not coplanar that a vector equation such as

$$\rho = \varpi,$$

or
$$x\alpha + y\beta + z\gamma = \xi\alpha + \eta\beta + \zeta\gamma,$$

necessitates the three numerical equations

$$x = \xi, \qquad y = \eta, \qquad z = \zeta.$$

For, if $\alpha, \beta, \gamma$ be coplanar (§ 24), a condition of the following form must hold

$$\gamma = a\alpha + b\beta.$$

Hence
$$\rho = (x + za)\,\alpha + (y + zb)\,\beta,$$
$$\varpi = (\xi + \zeta a)\,\alpha + (\eta + \zeta b)\,\beta,$$

and the equation
$$\rho = \varpi$$

now requires only the *two* numerical conditions

$$x + za = \xi + \zeta a, \qquad y + zb = \eta + \zeta b.$$

27. *The Commutative and Associative Laws hold in the combination of vectors by the signs* $+$ *and* $-$. It is obvious that, if we prove this for the sign $+$, it will be equally proved for $-$, because $-$ before a vector (§ 20) merely indicates that it is to be reversed before being considered positive.

Let $A, B, C, D$ be, in order, the corners of a parallelogram; we have, obviously,

$$\overline{AB} = \overline{DC}, \qquad \overline{AD} = \overline{BC}.$$

And
$$\overline{AB} + \overline{BC} = \overline{AC} = \overline{AD} + \overline{DC} = \overline{BC} + \overline{AB}.$$

Hence the commutative law is true for the addition of any two vectors, and is therefore generally true.

Again, whatever four points are represented by $A, B, C, D$, we have

$$\overline{AD} = \overline{AB} + \overline{BD} = \overline{AC} + \overline{CD},$$

or substituting their values for $\overline{AD}$, $\overline{BD}$, $\overline{AC}$ respectively, in these three expressions,

$$\overline{AB} + \overline{BC} + \overline{CD} = \overline{AB} + (\overline{BC} + \overline{CD}) = (\overline{AB} + \overline{BC}) + \overline{CD}.$$

And thus the truth of the associative law is evident.

28. The equation

$$\rho = x\beta,$$

where $\rho$ is the vector connecting a variable point with the origin, $\beta$ a definite vector, and $x$ an indefinite number, represents the straight line drawn from the origin parallel to $\beta$ (§ 22).

The straight line drawn from $A$, where $\overline{OA} = \alpha$, and parallel to $\beta$, has the equation

$$\rho = \alpha + x\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

In words, we may pass directly from $O$ to $P$ by the vector $\overline{OP}$ or $\rho$; or we may pass first to $A$, by means of $\overline{OA}$ or $\alpha$, and then to $P$ along a vector parallel to $\beta$ (§ 16).

Equation (1) is one of the many useful forms into which Quaternions enable us to throw the general equation of a straight line in space. As we have seen (§ 25) it is equivalent to *three* numerical equations; but, as these involve the indefinite quantity $x$, they are virtually equivalent to but *two*, as in ordinary Geometry of Three Dimensions.

29. A good illustration of this remark is furnished by the fact that the equation

$$\rho = y\alpha + x\beta,$$

which contains two indefinite quantities, is virtually equivalent to only one numerical equation. And it is easy to see that it represents the plane in which the lines $\alpha$ and $\beta$ lie; or the surface which is formed by drawing, through every point of $OA$, a line parallel to $OB$. In fact, the equation, as written, is simply § 24 in symbols.

And it is evident that the equation

$$\rho = \gamma + y\alpha + x\beta$$

is the equation of the plane passing through the extremity of $\gamma$, and parallel to $\alpha$ and $\beta$.

It will now be obvious to the reader that the equation

$$\rho = p_1\alpha_1 + p_2\alpha_2 + \ldots\ldots = \Sigma p\alpha,$$

where $\alpha_1$, $\alpha_2$, &c. are given vectors, and $p_1$, $p_2$, &c. numerical quantities, *represents a straight line* if $p_1$, $p_2$, &c. be linear functions of *one* indeterminate number; and a *plane*, if they be linear expressions containing *two* indeterminate numbers. Later (§ 31 ($l$)), this theorem will be much extended.

Again, the equation

$$\rho = x\alpha + y\beta + z\gamma$$

refers to *any* point whatever in space, provided $\alpha$, $\beta$, $\gamma$ are not coplanar. (*Ante*, § 23.)

30. The equation of the line joining any two points $A$ and $B$, where $\overline{OA} = \alpha$ and $\overline{OB} = \beta$, is obviously

$$\rho = \alpha + x(\beta - \alpha),$$

or

$$\rho = \beta + y(\alpha - \beta).$$

These equations are of course identical, as may be seen by putting $1 - y$ for $x$.

The first may be written

$$\rho + (x-1)\,\alpha - x\beta = 0\,;$$

or

$$p\rho + q\alpha + r\beta = 0,$$

subject to the condition $p + q + r = 0$ identically. That is—A homogeneous linear function of three vectors, equated to zero, expresses that the extremities of these vectors are in one straight line, *if the sum of the coefficients be identically zero.*

Similarly, the equation of the plane containing the extremities $A$, $B$, $C$ of the three non-coplanar vectors $\alpha$, $\beta$, $\gamma$ is

$$\rho = \alpha + x\,(\beta - \alpha) + y\,(\gamma - \beta),$$

where $x$ and $y$ are each indeterminate.

This may be written

$$p\rho + q\alpha + r\beta + s\gamma = 0,$$

with the identical relation

$$p + q + r + s = 0,$$

which is one form of the condition that four points may lie in one plane.

31. We have already the means of proving, in a very simple manner, numerous classes of propositions in plane and solid geometry. A very few examples, however, must suffice at this stage; since we have hardly, as yet, crossed the threshold of the subject, and are dealing with mere linear equations connecting two or more vectors, and even with them *we are restricted as yet to operations of mere addition.* We will give these examples with a painful minuteness of detail, which the reader will soon find to be necessary only for a short time, if at all.

(*a*) *The diagonals of a parallelogram bisect each other.*

Let $ABCD$ be the parallelogram, $O$ the point of intersection of its diagonals. Then

$$\overline{AO} + \overline{OB} = \overline{AB} = \overline{DC} = \overline{DO} + \overline{OC},$$

which gives

$$\overline{AO} - \overline{OC} = \overline{DO} - \overline{OB}.$$

The two vectors here equated are parallel to the diagonals respectively. Such an equation is, of course, absurd unless

(1) The diagonals are parallel, in which case the figure is not a parallelogram;

(2) $\overline{AO} = \overline{OC}$, and $\overline{DO} = \overline{OB}$, the proposition.

(*b*) *To shew that a triangle can be constructed, whose sides are parallel, and equal, to the bisectors of the sides of any triangle.*

Let $ABC$ be any triangle, $Aa$, $Bb$, $Cc$ the bisectors of the sides.

Then
$$\overline{Aa} = \overline{AB} + \overline{Ba} = \overline{AB} + \tfrac{1}{2}\overline{BC},$$
$$\overline{Bb} \quad - \quad - \quad - \quad = \overline{BC} + \tfrac{1}{2}\overline{CA},$$
$$\overline{Cc} \quad - \quad - \quad - \quad = \overline{CA} + \tfrac{1}{2}\overline{AB}.$$

Hence
$$\overline{Aa} + \overline{Bb} + \overline{Cc} = \tfrac{3}{2}(\overline{AB} + \overline{BC} + \overline{CA}) = 0;$$
which (§ 21) proves the proposition.

Also
$$\begin{aligned}\overline{Aa} &= \overline{AB} + \tfrac{1}{2}\overline{BC}\\ &= \overline{AB} - \tfrac{1}{2}(\overline{CA} + \overline{AB})\\ &= \tfrac{1}{2}(\overline{AB} - \overline{CA}) = \tfrac{1}{2}(\overline{AB} + \overline{AC}),\end{aligned}$$
results which are sometimes useful. They may be easily verified by producing $Aa$ to twice its length and joining the extremity with $B$.

(*b'*) *The bisectors of the sides of a triangle meet in a point, which trisects each of them.*

Taking $A$ as origin, and putting $\alpha, \beta, \gamma$ for vectors parallel, and equal, to the sides taken in order $BC$, $CA$, $AB$; the equation of $Bb$ is (§ 28 (1))
$$\rho = \gamma + x\left(\gamma + \frac{\beta}{2}\right) = (1 + x)\gamma + \frac{x}{2}\beta.$$

That of $Cc$ is, in the same way,
$$\rho = -(1 + y)\beta - \frac{y}{2}\gamma.$$

At the point $O$, where $Bb$ and $Cc$ intersect,
$$\rho = (1 + x)\gamma + \frac{x}{2}\beta = -(1 + y)\beta - \frac{y}{2}\gamma.$$

Since $\gamma$ and $\beta$ are not parallel, this equation gives
$$1 + x = -\frac{y}{2}, \text{ and } \frac{x}{2} = -(1 + y).$$

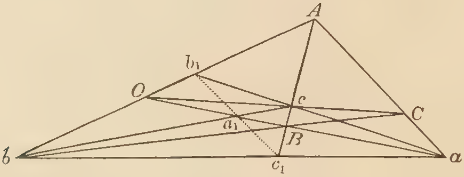
From these
$$x = y = -\tfrac{2}{3}.$$

Hence $\overline{AO} = \tfrac{1}{3}(\gamma - \beta) = \tfrac{2}{3}\overline{Aa}$. (See Ex. (*b*).)

This equation shews, being a vector one, that $Aa$ passes through $O$, and that $AO : Oa :: 2 : 1$.

(*c*) If
$$\overline{OA} = \alpha,$$
$$\overline{OB} = \beta,$$
$$\overline{OC} = l\alpha + m\beta,$$

*be three given co-planar vectors, c the intersection of AB, OC, and if the lines indicated in the figure be drawn, the points $a_1$, $b_1$, $c_1$ lie in a straight line.*



We see at once, by the process indicated in § 30, that

$$\overline{Oc} = \frac{l\alpha + m\beta}{l+m}, \qquad \overline{Ob} = \frac{l\alpha}{1-m}, \qquad \overline{Oa} = \frac{m\beta}{1-l}.$$

Hence we easily find

$$\overline{Oa_1} = -\frac{m\beta}{1-l-2m}, \qquad \overline{Ob_1} = -\frac{l\alpha}{1-2l-m}, \qquad \overline{Oc_1} = \frac{-l\alpha + m\beta}{m-l}.$$

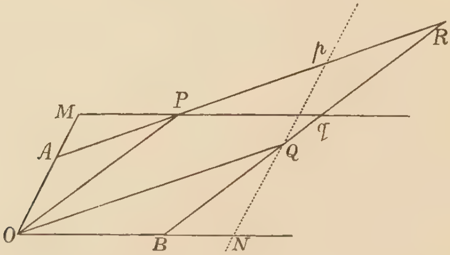
These give

$$-(1-l-2m)\,\overline{Oa_1} + (1-2l-m)\,\overline{Ob_1} - (m-l)\,\overline{Oc_1} = 0.$$

But $\quad -(1-l-2m) + (1-2l-m) - (m-l) = 0$ identically.

This, by § 30, proves the proposition.

(*d*) Let $\overline{OA} = \alpha$, $\overline{OB} = \beta$, be any two vectors. *If MP be a given line parallel to OB; and OQ, BQ, be drawn parallel to AP, OP respectively; the locus of Q is a straight line parallel to OA.*



Let $\qquad \overline{OM} = e\alpha.$

Then $\qquad \overline{AP} = \overline{e-1}\,\alpha + x\beta.$

Hence the equation of $OQ$ is

$$\rho = y(\overline{e-1}\alpha + x\beta);$$

and that of $BQ$ is $\rho = \beta + z(e\alpha + x\beta)$.

At $Q$ we have, therefore,

$$\left.\begin{aligned} xy &= 1 + zx, \\ y(e-1) &= ze. \end{aligned}\right\}$$

These give $xy = e$, and the equation of the locus of $Q$ is

$$\rho = e\beta + y'\alpha,$$

i.e. a straight line parallel to $OA$, drawn through $N$ in $OB$ produced, so that

$$ON : OB :: OM : OA.$$

COR. If $BQ$ meet $MP$ in $q$, $\overline{Pq} = \beta$; and if $AP$ meet $NQ$ in $p$, $\overline{Qp} = \alpha$.

Also, for the point $R$ we have $\overline{pR} = \overline{AP}$, $\overline{QR} = \overline{Bq}$.

Further, the locus of $R$ is a hyperbola, of which $MP$ and $NQ$ are the asymptotes. See, in this connection, § 31 ($k$) below.

Hence, *if from any two points, $A$ and $B$, lines be drawn intercepting a given length $Pq$ on a given line $Mq$; and if, from $R$ their point of intersection, $Rp$ be laid off $= PA$, and $RQ = qB$; $Q$ and $p$ lie on a fixed straight line, and the length of $Qp$ is constant.*

($e$) *To find the centre of inertia of any system of masses.*

If $\overline{OA} = \alpha$, $\overline{OB} = \alpha_1$, be the vector sides of any triangle, the vector from the vertex dividing the base $AB$ in $C$ so that

$$BC : CA :: m : m_1$$

is

$$\frac{m\alpha + m_1\alpha_1}{m + m_1}.$$

For $\overline{AB}$ is $\alpha_1 - \alpha$, and therefore $\overline{AC}$ is

$$\frac{m_1}{m + m_1}(\alpha_1 - \alpha).$$

Hence

$$\begin{aligned} \overline{OC} &= \overline{OA} + \overline{AC} \\ &= \alpha + \frac{m_1}{m + m_1}(\alpha_1 - \alpha) \\ &= \frac{m\alpha + m_1\alpha_1}{m + m_1}. \end{aligned}$$

This expression shews how to find the centre of inertia of two masses; $m$ at the extremity of $\alpha$, $m_1$ at that of $\alpha_1$. Introduce $m_2$

at the extremity of $\alpha_2$, then the vector of the centre of inertia of the three is, by a second application of the formula,

$$\frac{(m+m_1)\left(\frac{m\alpha+m_1\alpha_1}{m+m_1}\right)+m_2\alpha_2}{(m+m_1)+m_2}=\frac{m\alpha+m_1\alpha_1+m_2\alpha_2}{m+m_1+m_2}.$$

From this it is clear that, for any number of masses, expressed generally by $m$ at the extremity of the vector $\alpha$, the vector of the centre of inertia is

$$\beta=\frac{\Sigma\,(m\alpha)}{\Sigma\,(m)}.$$

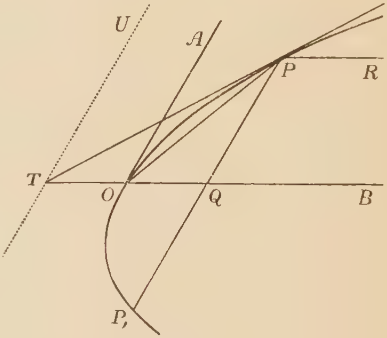
This may be written $\Sigma m\,(\alpha-\beta)=0.$

Now $\alpha_1-\beta$ is the vector of $m_1$ with respect to the centre of inertia. Hence the theorem, *If the vector of each element of a mass, drawn from the centre of inertia, be increased in length in proportion to the mass of the element, the sum of all these vectors is zero.*

(*f*) We see at once that the equation

$$\rho=\alpha t+\frac{\beta t^2}{2},$$

where $t$ is an indeterminate number, and $\alpha$, $\beta$ given vectors, represents a parabola. The origin, $O$, is a point on the curve, $\beta$ is parallel to the axis, i.e. is the diameter $OB$ drawn from the origin, and $\alpha$ is $OA$ the tangent at the origin. In the figure



$$\overline{QP}=\alpha t,\qquad \overline{OQ}=\frac{\beta t^2}{2}.$$

The secant joining the points where $t$ has the values $t$ and $t'$ is represented by the equation

$$\rho=\alpha t+\frac{\beta t^2}{2}+x\left(\alpha t'+\frac{\beta t'^2}{2}-\alpha t-\frac{\beta t^2}{2}\right)\quad(\S\ 30)$$

$$=\alpha t+\frac{\beta t^2}{2}+x\,(t'-t)\left\{\alpha+\beta\frac{t'+t}{2}\right\}.$$

Write $x$ for $x\,(t'-t)$ [which may have any value], then put $t'=t$, and the equation of the tangent at the point ($t$) is

$$\rho=\alpha t+\frac{\beta t^2}{2}+x\,(\alpha+\beta t).$$

In this put $x = -t$, and we have

$$\rho = -\frac{\beta t^2}{2},$$

or the intercept of the tangent on the diameter is equal in length to the abscissa of the point of contact, but has the opposite currency.

Otherwise: the tangent is parallel to the vector $\alpha + \beta t$ or $\alpha t + \beta t^2$ or $\frac{\beta t^2}{2} + \alpha t + \frac{\beta t^2}{2}$ or $\overline{OQ} + \overline{OP}$. But $\overline{TP} = \overline{TO} + \overline{OP}$, hence $\overline{TO} = \overline{OQ}$.

(*g*) Since the equation of any tangent to the parabola is

$$\rho = \alpha t + \frac{\beta t^2}{2} + x(\alpha + \beta t),$$

let us find the tangents which can be drawn from a given point. Let the vector of the point be

$$\rho = p\alpha + q\beta \quad (\S\ 24).$$

Since the tangent is to pass through this point, we have, as conditions to determine $t$ and $x$,

$$t + x = p,$$

$$\frac{t^2}{2} + xt = q;$$

by equating respectively the coefficients of $\alpha$ and $\beta$.

Hence $$t = p \pm \sqrt{p^2 - 2q}.$$

Thus, in general, *two* tangents can be drawn from a given point. These coincide if $$p^2 = 2q;$$

that is, if the vector of the point from which they are to be drawn is $$\rho = p\alpha + q\beta = p\alpha + \frac{p^2}{2}\beta,$$

i.e. if the point lies *on* the parabola. They are imaginary if $2q > p^2$, that is, if the point be

$$\rho = p\alpha + \left(\frac{p^2}{2} + r\right)\beta,$$

$r$ being *positive*. Such a point is evidently *within* the curve, as at $R$, where $\overline{OQ} = \frac{p^2}{2}\beta$, $\overline{QP} = p\alpha$, $\overline{PR} = r\beta$.

(*h*) Calling the values of $t$ for the two tangents found in (*g*) $t_1$ and $t_2$ respectively, it is obvious that the vector joining the points of contact is

$$\alpha t_1 + \frac{\beta t_1^2}{2} - \alpha t_2 - \frac{\beta t_2^2}{2},$$

which is parallel to $$\alpha + \beta\frac{t_1 + t_2}{2};$$

or, by the values of $t_1$ and $t_2$ in (*g*),

$$\alpha + p\beta.$$

Its direction, therefore, does not depend on $q$. In words, *If pairs of tangents be drawn to a parabola from points of a diameter produced, the chords of contact are parallel to the tangent at the vertex of the diameter.* This is also proved by a former result, for we must have $\overline{OT}$ for *each* tangent equal to $\overline{QO}$.

(*i*) The equation of the chord of contact, for the point whose vector is

$$\rho = p\alpha + q\beta,$$

is thus $$\rho = \alpha t_1 + \frac{\beta t_1^2}{2} + y(\alpha + p\beta).$$

Suppose this to pass always through the point whose vector is

$$\rho = a\alpha + b\beta.$$

Then we must have $$\left.\begin{aligned} t_1 + y &= a, \\ \frac{t_1^2}{2} + py &= b\,; \end{aligned}\right\}$$

or $$t_1 = p \pm \sqrt{p^2 - 2pa + 2b}.$$

Comparing this with the expression in (*g*), we have

$$q = pa - b;$$

that is, the point from which the tangents are drawn has the vector

$$\begin{aligned} \rho &= p\alpha + (pa - b)\beta \\ &= -b\beta + p(\alpha + a\beta), \text{ a straight line (§ 28 (1)).} \end{aligned}$$

The mere form of this expression contains the proof of the usual properties of the pole and polar in the parabola; but, for the sake of the beginner, we adopt a simpler, though equally general process.

Suppose $a = 0$. This merely restricts the pole to the particul

diameter to which we have referred the parabola. Then the pole is $Q$, where
$$\rho = b\beta;$$
and the polar is the line $TU$, for which
$$\rho = -b\beta + p\alpha.$$
Hence *the polar of any point is parallel to the tangent at the extremity of the diameter on which the point lies, and its intersection with that diameter is as far beyond the vertex as the pole is within, and vice versâ.*

($j$) As another example let us prove the following theorem. *If a triangle be inscribed in a parabola, the three points in which the sides are met by tangents at the angles lie in a straight line.*

Since $O$ is any point of the curve, we may take it as one corner of the triangle. Let $t$ and $t_1$ determine the others. Then, if $\varpi_1, \varpi_2, \varpi_3$ represent the vectors of the points of intersection of the tangents with the sides, we easily find

$$\varpi_1 = \frac{t_1^2}{2t_1 - t}\left(\alpha + \frac{t}{2}\beta\right),$$

$$\varpi_2 = \frac{t^2}{2t - t_1}\left(\alpha + \frac{t_1}{2}\beta\right),$$

$$\varpi_3 = \frac{tt_1}{t_1 + t}\alpha.$$

These values give

$$\frac{2t_1 - t}{t_1}\varpi_1 - \frac{2t - t_1}{t}\varpi_2 - \frac{t_1^2 - t^2}{tt_1}\varpi_3 = 0.$$

Also $$\frac{2t_1 - t}{t_1} - \frac{2t - t_1}{t} - \frac{t_1^2 - t^2}{tt_1} = 0 \text{ identically.}$$

Hence, by § 30, the proposition is proved.

($k$) Other interesting examples of this method of treating curves will, of course, suggest themselves to the student. Thus

$$\rho = \alpha \cos t + \beta \sin t$$

or $$\rho = \alpha x + \beta\sqrt{1 - x^2}$$

represents an ellipse, of which the given vectors $\alpha$ and $\beta$ are semi-conjugate diameters. If $t$ represent time, the radius-vector of this ellipse traces out equal areas in equal times. [We may anticipate so far as to write the following:—

$$2 \text{ Area} = T\int V\rho d\rho = TV\alpha\beta . \int dt;$$

which will be easily understood later.]

Again, $\rho = \alpha t + \frac{\beta}{t}$ or $\rho = \alpha \tan x + \beta \cot x$

evidently represents a hyperbola referred to its asymptotes. [If $t$ represent time, the sectorial area traced out is proportional to $\log t$, taken between proper limits.]

Thus, also, the equation

$$\rho = \alpha (t + \sin t) + \beta \cos t$$

in which $\alpha$ and $\beta$ are of equal lengths, and at right angles to one another, represents a cycloid. The origin is at the middle point of the axis $(2\beta)$ of the curve. [It may be added that, if $t$ represent *time*, this equation shews the motion of the tracing point, provided the generating circle rolls uniformly, revolving at the rate of a radian per second.]

When the lengths of $\alpha$, $\beta$ are not equal, this equation gives the cycloid distorted by elongation of its ordinates or abscissae :—*not* a trochoid. The equation of a trochoid may be written

$$\rho = \alpha (et + \sin t) + \beta \cos t,$$

$e$ being greater or less than 1 as the curve is prolate or curtate. The lengths of $\alpha$ and $\beta$ are still taken as equal.

But, so far as we have yet gone with the explanation of the calculus, as we are not prepared to determine the lengths or inclinations of vectors, we can investigate only a very limited class of the properties of curves, represented by such equations as those above written.

(*l*) We may now, in extension of the statement in § 29, make the obvious remark that

$$\rho = \Sigma p\alpha$$

(where, as in § 23, the number of vectors, $\alpha$, can always be reduced to *three*, at most) is the equation of a *curve* in space, if the numbers $p_1$, $p_2$, &c. are functions of *one* indeterminate. In such a case the equation is sometimes written

$$\rho = \phi (t).$$

But, if $p_1$, $p_2$, &c. be functions of *two* indeterminates, the locus of the extremity of $\rho$ is a *surface*; whose equation is sometimes written

$$\rho = \phi (t, u).$$

[It may not be superfluous to call the reader's attention to the fact that, in these equations, $\phi (t)$ or $\phi (t, u)$ is *necessarily* a vector expression, since it is equated to a vector, $\rho$.]

(*m*) Thus the equation

$$\rho = \alpha \cos t + \beta \sin t + \gamma t \qquad \text{.....................(1)}$$

belongs to a helix, while

$$\rho = \alpha \cos t + \beta \sin t + \gamma u \qquad \text{..................(2)}$$

represents a cylinder whose generating lines are parallel to $\gamma$, and whose base is the ellipse

$$\rho = \alpha \cos t + \beta \sin t.$$

The helix above lies wholly on this cylinder.

Contrast with (2) the equation

$$\rho = u (\alpha \cos t + \beta \sin t + \gamma) \qquad \text{.....................(3)}$$

which represents a cone of the second degree:—made up, in fact, of all lines drawn from the origin to the ellipse

$$\rho = \alpha \cos t + \beta \sin t + \gamma.$$

If, however, we write

$$\rho = u (\alpha \cos t + \beta \sin t + \gamma t),$$

we form the equation of the transcendental cone whose vertex is at the origin, and on which lies the helix (1).

In general

$$\rho = u\phi (t)$$

is the cone whose vertex is the origin, and on which lies the curve

$$\rho = \phi (t);$$

while

$$\rho = \phi (t) + u\alpha$$

is a cylinder, with generating lines parallel to $\alpha$, standing on the same curve as base.

Again,

$$\rho = p\alpha + q\beta + r\gamma$$

with a condition of the form

$$ap^2 + bq^2 + cr^2 = 1$$

belongs to a central surface of the second order, of which $\alpha, \beta, \gamma$ are the directions of conjugate diameters. If $a, b, c$ be all positive, the surface is an ellipsoid.

32. In Example (*f*) above we performed an operation equivalent to the differentiation of a vector with reference to a single *numerical* variable of which it was given as an explicit function. As this process is of very great use, especially in quaternion investigations connected with the motion of a particle or point; and as it will afford us an opportunity of making a preliminary step towards

overcoming the novel difficulties which arise in quaternion differentiation; we will devote a few sections to a more careful, though very elementary, exposition of it.

33. It is a striking circumstance, when we consider the way in which Newton's original methods in the Differential Calculus have been decried, to find that Hamilton was *obliged* to employ them, and not the more modern forms, in order to overcome the characteristic difficulties of quaternion differentiation. Such a thing as *a differential coefficient has absolutely no meaning in quaternions*, except in those special cases in which we are dealing with degraded quaternions, such as numbers, Cartesian coördinates, &c. But a quaternion expression has always a *differential*, which is, simply, what Newton called a *fluxion*.

As with the Laws of Motion, the basis of Dynamics, so with the foundations of the Differential Calculus; we are gradually coming to the conclusion that Newton's system is the best after all.

34. Suppose $\rho$ to be the vector of a curve in space. Then, generally, $\rho$ may be expressed as the sum of a number of terms, each of which is a multiple of a constant vector by a function of some *one* indeterminate; or, as in § 31 ($l$), if $P$ be a point on the curve,

$$\overline{OP} = \rho = \phi(t).$$

And, similarly, if $Q$ be *any other* point on the curve,

$$\overline{OQ} = \rho_1 = \rho + \delta\rho = \phi(t_1) = \phi(t + \delta t),$$

where $\delta t$ is *any number whatever.*

The vector-chord $\overline{PQ}$ is therefore, rigorously,

$$\delta\rho = \rho_1 - \rho = \phi(t + \delta t) - \phi t.$$

35. It is obvious that, in the present case, *because the vectors involved in $\phi$ are constant, and their numerical multipliers alone vary*, the expression $\phi(t + \delta t)$ is, by Taylor's Theorem, equivalent to

$$\phi(t) + \frac{d\phi(t)}{dt}\delta t + \frac{d^2\phi(t)}{dt^2}\frac{(\delta t)^2}{1.2} + \ldots\ldots$$

Hence,
$$\delta\rho = \frac{d\phi(t)}{dt}\delta t + \frac{d^2\phi(t)}{dt^2}\frac{(\delta t)^2}{1.2} + \&c.$$

And we are thus entitled to write, when $\delta t$ has been made indefinitely small,

$$\text{Limit}\left(\frac{\delta\rho}{\delta t}\right)_{\delta t=0} = \frac{d\rho}{dt} = \frac{d\phi(t)}{dt} = \phi'(t).$$

In such a case as this, then, we are permitted to differentiate, or to form the differential coefficient of, a vector, according to the ordinary rules of the Differential Calculus. But great additional insight into the process is gained by applying Newton's method.

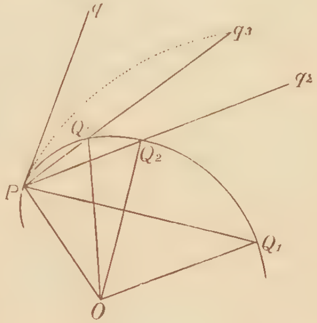
36. Let $\overline{OP}$ be

$$\rho = \phi(t),$$

and $\overline{OQ}_1$

$$\rho_1 = \phi(t + dt),$$

where $dt$ is any number whatever.



The number $t$ may here be taken as representing *time*, i.e. we may suppose a point to move along the curve in such a way that the value of $t$ for the vector of the point $P$ of the curve denotes the interval which has elapsed (since a fixed epoch) when the moving point has reached the extremity of that vector. If, then, $dt$ represent any interval, finite or not, we see that

$$\overline{OQ}_1 = \phi(t + dt)$$

will be the vector of the point after the additional interval $dt$.

But this, in general, gives us little or no information as to the velocity of the point at $P$. We shall get a better approximation by halving the interval $dt$, and finding $Q_2$, where $\overline{OQ}_2 = \phi(t + \frac{1}{2}dt)$, as the position of the moving point at that time. Here the vector virtually described in $\frac{1}{2}dt$ is $\overline{PQ}_2$. To find, on this supposition, the vector described in $dt$, we must double $\overline{PQ}_2$, and we find, as a second approximation to the vector which the moving point would have described in time $dt$, if it had moved for that period in the direction and with the velocity it had at $P$,

$$\begin{aligned}\overline{Pq}_2 = 2\,\overline{PQ}_2 &= 2(\overline{OQ}_2 - \overline{OP}) \\ &= 2\{\phi(t + \tfrac{1}{2}dt) - \phi(t)\}.\end{aligned}$$

The next approximation gives

$$\begin{aligned}\overline{Pq}_3 = 3\overline{PQ}_3 &= 3(\overline{OQ}_3 - \overline{OP}) \\ &= 3\{\phi(t + \tfrac{1}{3}dt) - \phi(t)\}.\end{aligned}$$

And so on, each step evidently leading us nearer the sought truth. Hence, to find the vector which would have been described in time $dt$ had the circumstances of the motion at $P$ remained undisturbed, we must find the value of

$$d\rho = \overline{Pq} = \mathcal{L}_{x=\infty}\, x \left\{\phi\left(t + \frac{1}{x}dt\right) - \phi(t)\right\}.$$

We have seen that in this particular case we may use Taylor's Theorem. We have, therefore,

$$d\rho = \mathcal{L}_{x=\infty}\, x \left\{ \phi'(t) \frac{1}{x} dt + \phi''(t) \frac{1}{x^2} \frac{(dt)^2}{1\,.\,2} + \&c. \right\}$$

$$= \phi'(t)\, dt.$$

And, if we choose, we may now write

$$\frac{d\rho}{dt} = \phi'(t).$$

37. But it is to be most particularly remarked that in the whole of this investigation no regard whatever has been paid to the magnitude of $dt$. The question which we have now answered may be put in the form—*A point describes a given curve in a given manner. At any point of its path its motion suddenly ceases to be accelerated. What space will it describe in a definite interval?* As Hamilton well observes, this is, for a planet or comet, the case of a 'celestial Atwood's machine.'

38. If we suppose the variable, in terms of which $\rho$ is expressed, to be the arc, $s$, of the curve measured from some fixed point, we find as before

$$d\rho = \phi'(s)\, ds.$$

From the very nature of the question it is obvious that the length of $d\rho$ must in this case be $ds$, so that $\phi'(s)$ is necessarily a unit-vector. This remark is of importance, as we shall see later; and it may therefore be useful to obtain afresh the above result without any reference to time or velocity.

39. Following strictly the process of Newton's VIIth Lemma, let us describe on $Pq_2$ an arc similar to $PQ_2$, and so on. Then obviously, as the subdivision of $ds$ is carried farther, the new arc (whose length is always $ds$) more and more nearly (and without limit) coincides with the line which expresses the corresponding approximation to $d\rho$.

40. As additional examples let us take some well-known *plane* curves; and first the hyperbola (§ 31 ($k$))

$$\rho = \alpha t + \frac{\beta}{t}.$$

Here

$$d\rho = \left(\alpha - \frac{\beta}{t^2}\right) dt.$$

This shews that the tangent is parallel to the vector

$$\alpha t - \frac{\beta}{t}.$$

In words, *if the vector (from the centre) of a point in a hyperbola be one diagonal of a parallelogram, two of whose sides coincide with the asymptotes, the other diagonal is parallel to the tangent at the point, and cuts off a constant area from the space between the asymptotes.* (For the sides of this triangular area are $t$ times the length of $\alpha$, and $1/t$ times the length of $\beta$, respectively; the angle between them being constant.)

Next, take the cycloid, as in § 31 ($k$),

$$\rho = \alpha (t + \sin t) + \beta \cos t.$$

We have

$$d\rho = \{\alpha (1 + \cos t) - \beta \sin t\}\, dt.$$

At the vertex

$$t = 0, \quad \cos t = 1, \quad \sin t = 0, \quad \text{and } d\rho = 2\alpha dt.$$

At a cusp

$$t = \pi, \quad \cos t = -1, \quad \sin t = 0, \quad \text{and } d\rho = 0.$$

This indicates that, at the cusp, the tracing point is (instantaneously) at rest. To find the direction of the tangent, and the form of the curve in the vicinity of the cusp, put $t = \pi + \tau$, where powers of $\tau$ above the second are omitted. We have

$$d\rho = \beta\tau dt + \frac{\alpha\tau^2}{2} dt,$$

so that, at the cusp, the tangent is parallel to $\beta$. By making the same substitution in the expression for $\rho$, we find that the part of the curve near the cusp is a semicubical parabola,

$$\rho = \alpha (\pi + \tau^3/6) - \beta (1 - \tau^2/2);$$

or, if the origin be shifted to the cusp ($\rho = \pi\alpha - \beta$),

$$\rho = \alpha\tau^3/6 + \beta\tau^2/2.$$

41. Let us reverse the first of these questions, and *seek the envelop of a line which cuts off from two fixed axes a triangle of constant area.*

If the axes be in the directions of $\alpha$ and $\beta$, the intercepts may evidently be written $\alpha t$ and $\frac{\beta}{t}$. Hence the equation of the line is (§ 30)

$$\rho = \alpha t + x\left(\frac{\beta}{t} - \alpha t\right).$$

The condition of envelopment is, obviously, (see Chap. IX.)

$$d\rho = 0.$$

This giveso $$0 = \left\{\alpha - x\left(\frac{\beta}{t^2} + \alpha\right)\right\} dt + \left(\frac{\beta}{t} - \alpha t\right) dx^*.$$

Hence $$(1 - x)\, dt - t\, dx = 0,$$

and $$-\frac{x}{t^2}\, dt + \frac{dx}{t} = 0.$$

From these, at once, $x = \frac{1}{2}$, since $dx$ and $dt$ are indeterminate. Thus the equation of the envelop is

$$\rho = \alpha t + \tfrac{1}{2}\left(\frac{\beta}{t} - \alpha t\right)$$

$$= \tfrac{1}{2}\left(\alpha t + \frac{\beta}{t}\right),$$

the hyperbola as before; $\alpha$, $\beta$ being portions of its asymptotes.

42. It may assist the student to a thorough comprehension of the above process, if we put it in a slightly different form. Thus the equation of the enveloping line may be written

$$\rho = \alpha t(1 - x) + \beta\frac{x}{t},$$

which gives $$d\rho = 0 = \alpha d\{t(1 - x)\} + \beta d\left(\frac{x}{t}\right).$$

Hence, as $\alpha$ is not parallel to $\beta$, we must have

$$d\{t(1 - x)\} = 0, \qquad d\left(\frac{x}{t}\right) = 0;$$

and these are, when expanded, the equations we obtained in the preceding section.

43. For farther illustration we give a solution not directly employing the differential calculus. The equations of any two of the enveloping lines are

---

[*] Here we have opportunity for a remark (very simple indeed, but) of the utmost importance. *We are not to equate separately to zero the coefficients of dt* and $dx$; for we must remember that this equation is of the form

$$0 = p\alpha + q\beta,$$

where $p$ and $q$ are numbers; and that, so long as $\alpha$ and $\beta$ are actual and non-parallel vectors, the existence of such an equation requires (§ 24)

$$p = 0 \qquad q = 0.$$

$$\rho = \alpha t + x\left(\frac{\beta}{t} - \alpha t\right),$$

$$\rho = \alpha t_1 + x_1\left(\frac{\beta}{t_1} - \alpha t_1\right),$$

$t$ and $t_1$ being given, while $x$ and $x_1$ are indeterminate.

At the point of intersection of these lines we have (§ 26),

$$\left.\begin{array}{c} t(1-x) = t_1(1-x_1), \\ \dfrac{x}{t} = \dfrac{x_1}{t_1} \end{array}\right\}.$$

These give, by eliminating $x_1$,

$$t(1-x) = t_1\left(1 - \frac{t_1}{t}x\right),$$

or
$$x = \frac{t}{t_1 + t}.$$

Hence the vector of the point of intersection is

$$\rho = \frac{\alpha t t_1 + \beta}{t_1 + t},$$

and thus, for the ultimate intersections, where $\mathcal{L}\dfrac{t_1}{t} = 1$,

$$\rho = \tfrac{1}{2}\left(\alpha t + \frac{\beta}{t}\right) \text{ as before.}$$

Cor. If. instead of the *ultimate* intersections, we consider the intersections of pairs of these lines related by some law, we obtain useful results. Thus let

$$tt_1 = 1,$$

$$\rho = \frac{\alpha + \beta}{t + \dfrac{1}{t}};$$

or the intersection lies in the diagonal of the parallelogram on $\alpha, \beta$.

If $t_1 = mt$, where $m$ is constant,

$$\rho = \frac{mt\alpha + \dfrac{\beta}{t}}{m+1}.$$

But we have also $x = \dfrac{1}{m+1}$.

Hence *the locus of a point which divides in a given ratio a line cutting off a given area from two fixed axes, is a hyperbola of which these axes are the asymptotes.*

If we take either

$$tt_1(t+t_1) = \text{constant, or } \frac{t^2t_1^{\,2}}{t+t_1} = \text{constant},$$

the locus is a parabola; and so on.

It will be excellent practice for the student, at this stage, to work out in detail a number of similar questions relating to the envelop of, or the locus of the intersection of selected pairs from, a series of lines drawn according to a given law. And the process may easily be extended to planes. Thus, for instance, we may form the general equation of planes which cut off constant tetrahedra from the axes of cöordinates. Their envelop is a surface of the third degree whose equation may be written

$$\rho = x\alpha + y\beta + z\gamma\,;$$

where $$xyz = a^3.$$

Again, find the locus of the point of intersection of three of this group of planes, such that the first intercepts on $\beta$ and $\gamma$, the second on $\gamma$ and $\alpha$, the third on $\alpha$ and $\beta$, lengths all equal to one another, &c. But we must not loiter with such simple matters as these.

44. The reader who is fond of Anharmonic Ratios and Transversals will find in the early chapters of Hamilton's *Elements of Quaternions* an admirable application of the composition of vectors to these subjects. The Theory of Geometrical Nets, in a plane, and in space, is there very fully developed; and the method is shewn to include, as particular cases, the corresponding processes of Grassmann's *Ausdehnungslehre* and Möbius' *Barycentrische Calcul.* Some very curious investigations connected with curves and surfaces of the second and third degrees are also there founded upon the composition of vectors.

## EXAMPLES TO CHAPTER I.

1. The lines which join, towards the same parts, the extremities of two equal and parallel lines are themselves equal and parallel. (*Euclid*, I. xxxiii.)

2. Find the vector of the middle point of the line which joins the middle points of the diagonals of any quadrilateral, plane or

gauche, the vectors of the corners being given; and so prove that this point is the mean point of the quadrilateral.

If two opposite sides be divided proportionally, and two new quadrilaterals be formed by joining the points of division, the mean points of the three quadrilaterals lie in a straight line.

Shew that the mean point may also be found by bisecting the line joining the middle points of a pair of opposite sides.

3. Verify that the property of the coefficients of three vectors whose extremities are in a line (§ 30) is not interfered with by altering the origin.

4. If two triangles $ABC$, $abc$, be so situated in space that $Aa$, $Bb$, $Cc$ meet in a point, the intersections of $AB$, $ab$, of $BC$, $bc$, and of $CA$, $ca$, lie in a straight line.

5. Prove the converse of 4, i. e. if lines be drawn, one in each of two planes, from any three points in the straight line in which these planes meet, the two triangles thus formed are sections of a common pyramid.

6. If five quadrilaterals be formed by omitting in succession each of the sides of any pentagon, the lines bisecting the diagonals of these quadrilaterals meet in a point. (H. Fox Talbot.)

7. Assuming, as in § 7, that the operator

$$\cos\theta + \sqrt{-1}\sin\theta$$

turns any radius of a given circle through an angle $\theta$ in the positive direction of rotation, without altering its length, deduce the ordinary formulæ for $\cos(A+B)$, $\cos(A-B)$, $\sin(A+B)$, and $\sin(A-B)$, in terms of sines and cosines of $A$ and $B$.

8. If two tangents be drawn to a hyperbola, the line joining the centre with their point of intersection bisects the lines joining the points where the tangents meet the asymptotes: and the secant through the points of contact bisects the intercepts on the asymptotes.

9. Any two tangents, limited by the asymptotes, divide each other proportionally.

10. If a chord of a hyperbola be one diagonal of a parallelogram whose sides are parallel to the asymptotes, the other diagonal passes through the centre.

11. Given two points $A$ and $B$, and a plane, $C$. Find the locus of $P$, such that if $AP$ cut $C$ in $Q$, and $BP$ cut $C$ in $R$, $\overline{QR}$ may be a given vector.

12. Shew that $\rho = x^2\alpha + y^2\beta + (x+y)^2\gamma$
is the equation of a cone of the second degree, and that its section by the plane

$$\rho = \frac{p\alpha + q\beta + r\gamma}{p+q+r}$$

is an ellipse which touches, at their middle points, the sides of the triangle of whose corners $\alpha$, $\beta$, $\gamma$ are the vectors. (Hamilton, *Elements*, p. 96.)

13. The lines which divide, proportionally, the pairs of opposite sides of a gauche quadrilateral, are the generating lines of a hyperbolic paraboloid. (*Ibid.* p. 97.)

14. Shew that $\rho = x^3\alpha + y^3\beta + z^3\gamma,$
where $x + y + z = 0,$
represents a cone of the third order, and that its section by the plane

$$\rho = \frac{p\alpha + q\beta + r\gamma}{p+q+r}$$

is a cubic curve, of which the lines

$$\rho = \frac{p\alpha + q\beta}{p+q}, \text{ \&c.}$$

are the asymptotes and the three (real) tangents of inflection. Also that the mean point of the triangle formed by these lines is a conjugate point of the curve. Hence that the vector $\alpha + \beta + \gamma$ is a conjugate ray of the cone. (*Ibid.* p. 96.)

# CHAPTER II.

## PRODUCTS AND QUOTIENTS OF VECTORS.

45. We now come to the consideration of questions in which the Calculus of Quaternions differs entirely from any previous mathematical method; and here we shall get an idea of what a Quaternion is, and whence it derives its name. These questions are fundamentally involved in the novel use of the symbols of multiplication and division. And the simplest introduction to the subject seems to be the consideration of the quotient, or ratio, of two vectors.

46. If the given vectors be parallel to each other, we have already seen (§ 22) that either may be expressed as a *numerical* multiple of the other; the multiplier being simply the ratio of their lengths, taken positively if they have similar currency, negatively if they run opposite ways.

47. If they be not parallel, let $\overline{OA}$ and $\overline{OB}$ be drawn parallel and equal to them from any point $O$; and the question is reduced to finding the value of the ratio of two vectors drawn from the same point. Let us first find *upon how many distinct numbers this ratio depends.*

We may suppose $\overline{OA}$ to be changed into $\overline{OB}$ by the following successive processes.

1st. Increase or diminish the length of $\overline{OA}$ till it becomes equal to that of $\overline{OB}$. For this only *one* number is required, viz. the ratio of the lengths of the two vectors. As Hamilton remarks, this is a positive, or rather a *signless*, number.

2nd. Turn $\overline{OA}$ about $O$, in the common plane of the two vectors, until its direction coincides with that of $\overline{OB}$, and (remembering the effect of the first operation) we see that the two vectors now coincide or become identical. To specify this operation *three* numbers are required, viz. *two* angles (such as node and inclination in the case of a planet's orbit) to fix the plane in which the rotation takes place, and *one* angle for the amount of this rotation.

Thus it appears that the ratio of two vectors, or the multiplier required to change one vector into another, in general depends upon *four* distinct numbers, whence the name QUATERNION.

A quaternion $q$ is thus *defined* as expressing a relation

$$\beta = q\alpha$$

between two vectors $\alpha$, $\beta$. By what precedes, the vectors $\alpha$, $\beta$, which serve for the definition of a given quaternion, must be in a given plane, at a given inclination to each other, and with their lengths in a given ratio; but it is to be noticed that they may be *any* two such vectors. [*Inclination* is understood to include sense, or currency, of rotation from $\alpha$ to $\beta$.]

The particular case of perpendicularity of the two vectors, where their quotient is a vector perpendicular to their plane, is fully considered below; §§ 64, 65, 72, &c.

48. It is obvious that the operations just described may be performed, with the same result, in the opposite order, being perfectly independent of each other. Thus it appears that a quaternion, considered as the factor or agent which changes one definite vector into another, may itself be decomposed into two factors of which the order is immaterial.

The *stretching* factor, or that which performs the first operation in § 47, is called the TENSOR, and is denoted by prefixing $T$ to the quaternion considered.

The *turning* factor, or that corresponding to the second operation in § 47, is called the VERSOR, and is denoted by the letter $U$ prefixed to the quaternion.

49. Thus, if $\overline{OA} = \alpha$, $\overline{OB} = \beta$, and if $q$ be the quaternion which changes $\alpha$ to $\beta$, we have

$$\beta = q\alpha,$$

which we may write in the form

$$\frac{\beta}{\alpha} = q, \text{ or } \beta\alpha^{-1} = q,$$

if we agree to *define* that

$$\frac{\beta}{\alpha} \cdot \alpha = \beta\alpha^{-1} \cdot \alpha = \beta.$$

Here it is to be particularly noticed that we write $q$ *before* $\alpha$ to signify that $\alpha$ is multiplied by (or operated on by) $q$, not $q$ multiplied by $\alpha$.

This remark is of extreme importance in quaternions, for, as we shall soon see, the Commutative Law does not generally apply to the factors of a product.

We have also, by §§ 47, 48,

$$q = Tq \, . \, Uq = Uq \, . \, Tq,$$

where, as before, $Tq$ depends merely on the relative lengths of $\alpha$ and $\beta$, and $Uq$ depends solely on their directions.

Thus, if $\alpha_1$ and $\beta_1$ be vectors of unit length parallel to $\alpha$ and $\beta$ respectively,

$$T\frac{\beta_1}{\alpha_1} = T\beta_1/T\alpha_1 = 1, \qquad U\frac{\beta_1}{\alpha_1} = U\beta_1/U\alpha_1 = U\frac{\beta}{\alpha}.$$

As will soon be shewn, when $\alpha$ is perpendicular to $\beta$, i.e. when the versor of the quotient is quadrantal, it is a unit-vector.

50. We must now carefully notice that the quaternion which is the quotient when $\beta$ is divided by $\alpha$ in no way depends upon the *absolute* lengths, or directions, of these vectors. Its value will remain unchanged if we substitute for them any other pair of vectors which

(1) have their lengths in the same ratio,

(2) have their common plane the same or parallel,

and (3) make the same angle with each other.

Thus in the annexed figure

$$\frac{\overline{O_1B_1}}{\overline{O_1A_1}} = \frac{\overline{OB}}{\overline{OA}}$$

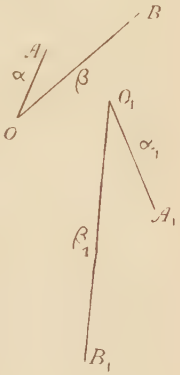
if, and only if,

(1) $\dfrac{O_1B_1}{O_1A_1} = \dfrac{OB}{OA}$,

(2) plane $AOB$ parallel to plane $A_1O_1B_1$,

(3) $\angle AOB = \angle A_1O_1B_1$.

[Equality of angles is understood to include concurrency of rotation. Thus in the annexed figure the rotation about an axis drawn upwards from the plane is negative (or clock-wise) from $OA$ to $OB$, and also from $O_1A_1$ to $O_1B_1$.]



It thus appears that if

$$\beta = q\alpha, \;\; \delta = q\gamma,$$

the vectors $\alpha$, $\beta$, $\gamma$, $\delta$ are parallel to one plane, and may be represented (in a highly extended sense) as *proportional* to one another, thus:—

$$\beta : \alpha = \delta : \gamma.$$

And it is clear from the previous part of this section that this may be written not only in the form

$$\alpha : \beta = \gamma : \delta$$

but also in either of the following forms:—

$$\gamma : \alpha = \delta : \beta.$$

$$\alpha : \gamma = \beta : \delta.$$

While these proportions are true as equalities of ratios, they do not usually imply equalities of products.

Thus, as the first of these was equivalent to the equation

$$\frac{\beta}{\alpha} = \frac{\delta}{\gamma} = q, \text{ or } \beta\alpha^{-1} = \delta\gamma^{-1} = q;$$

the following three imply separately, (see next section)

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = q^{-1}, \;\; \frac{\gamma}{\alpha} = \frac{\delta}{\beta} = r, \;\; \frac{\alpha}{\gamma} = \frac{\beta}{\delta} = r^{-1},$$

or, if we please,

$$\alpha\beta^{-1} = \gamma\delta^{-1} = q^{-1}, \quad \gamma\alpha^{-1} = \delta\beta^{-1} = r, \quad \alpha\gamma^{-1} = \beta\delta^{-1} = r^{-1};$$

where $r$ is a *new* quaternion, which has not necessarily anything (except its plane), in common with $q$.

But here great caution is requisite, for we are *not* entitled to conclude from these that

$$\alpha\delta = \beta\gamma, \text{ \&c.}$$

This point will be fully discussed at a later stage. Meanwhile we may merely *state* that from

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta}, \text{ or } \frac{\beta}{\alpha} = \frac{\delta}{\gamma},$$

we are entitled to deduce a number of equivalents such as

$$\alpha\beta^{-1}\delta = \gamma, \text{ or } \alpha = \gamma\delta^{-1}\beta, \text{ or } \beta^{-1}\delta = \alpha^{-1}\gamma, \text{ \&c.}$$

51. The *Reciprocal* of a quaternion $q$ is defined by the equation

$$\frac{1}{q}q = q^{-1}q = 1 = q\frac{1}{q} = qq^{-1}.$$

Hence if $$\frac{\beta}{\alpha} = q, \text{ or}$$

$$\beta = q\alpha,$$

we must have $$\frac{\alpha}{\beta} = \frac{1}{q} = q^{-1}.$$

For this gives $$\frac{\alpha}{\beta} . \beta = q^{-1} . q\alpha,$$

and each member of the equation is evidently equal to $\alpha$.

Or thus :—

$$\beta = q\alpha$$

Operate *by* $q^{-1}$,

$$q^{-1}\beta = \alpha.$$

Operate *on* $\beta^{-1}$,

$$q^{-1} = \alpha\beta^{-1} = \frac{\alpha}{\beta}.$$

Or, we may reason thus :—since $q$ changes $\overline{OA}$ to $OB$, $q^{-1}$ must change $\overline{OB}$ to $\overline{OA}$, and is therefore expressed by $\frac{\alpha}{\beta}$ (§ 49).

The tensor of the reciprocal of a quaternion is therefore the reciprocal of the tensor; and the versor differs merely by the *reversal* of its representative angle. The versor, it must be remembered, gives the plane and angle of the turning—it has nothing to do with the extension.

[*Remark.* In §§ 49—51, above, we had such expressions as $\frac{\beta}{\alpha} = \beta\alpha^{-1}$. We have also met with $\alpha^{-1}\beta$. Cayley suggests that this also may be written in the ordinary fractional form by employing the following distinctive notation :—

$$\frac{\beta}{\alpha} = \beta\alpha^{-1} = \frac{\beta\rfloor}{\lfloor\alpha}, \quad \alpha^{-1}\beta = \frac{\lfloor\beta}{\alpha\rfloor}.$$

(It might, perhaps, be even simpler to use the *solidus* as recommended by Stokes, along with an obviously correlative type :—thus,

$$\frac{\beta}{\alpha} = \beta\alpha^{-1} = \beta/\alpha, \quad \alpha^{-1}\beta = \alpha\backslash\beta.)$$

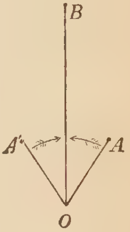
I have found such notations occasionally convenient for private work, but I hesitate to introduce changes unless they are absolutely required. See remarks on this point towards the end of the *Preface to the Second Edition*—reprinted above.]

52. The *Conjugate* of a quaternion $q$, written $Kq$, has the same tensor, plane, and angle, only the angle is taken the reverse way; or the versor of the conjugate is the reciprocal of the versor of the quaternion, or (what comes to the same thing) the versor of the reciprocal.

Thus, if $OA$, $OB$, $OA'$, lie in one plane, and if

$OA' = OA$, and $\angle A'OB = \angle BOA$, we have

$$\frac{\overline{OB}}{\overline{OA}} = q, \text{ and } \frac{\overline{OB}}{\overline{OA'}} = \text{conjugate of } q = Kq.$$



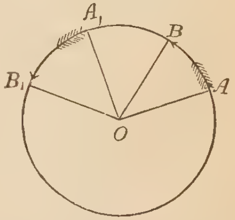
By last section we see that

$$Kq = (Tq)^2 q^{-1}.$$

Hence $$qKq = Kq \,.\, q = (Tq)^2.$$

This proposition is obvious, if we recollect that the tensors of $q$ and $Kq$ are equal, and that the versors are such that either *annuls* the effect of the other; while the order of their application is indifferent. The joint effect of these factors is therefore merely to multiply twice over by the common tensor.

53. It is evident from the results of § 50 that, if $\alpha$ and $\beta$ be of equal length, they may be treated as of unit-length so far as their quaternion quotient is concerned. This quotient is therefore a versor (the tensor being unity) and may be represented indifferently by any one of an infinite number of concurrent arcs of given length lying on the circumference of a circle, of which the two vectors are radii. This is of considerable importance in the proofs which follow.



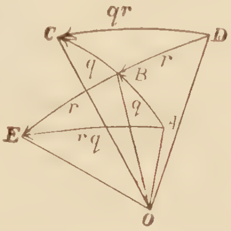
Thus the versor $\frac{\overline{OB}}{\overline{OA}}$ may be represented in magnitude, plane, and currency of rotation (§ 50) by the arc $AB$, which may in this extended sense be written $\overset{\frown}{AB}$.

And, similarly, the versor $\frac{\overline{OB_1}}{\overline{OA_1}}$ is represented by $\overset{\frown}{A_1B_1}$; which is equal to (and concurrent with) $\overset{\frown}{AB}$ if

$$\angle A_1OB_1 = \angle AOB,$$

i.e. if the versors are *equal*, in the quaternion meaning of the word.

54. By the aid of this process, when a versor is represented as an arc of a great circle on the unit-sphere, we can easily prove that *quaternion multiplication is not generally commutative.*



Thus let $q$ be the versor $\overset{\frown}{AB}$ or $\frac{OB}{OA}$, where $O$ is the centre of the sphere. Take $\overset{\frown}{BC}=\overset{\frown}{AB}$, (which, it must be remembered, makes the points $A$, $B$, $C$, lie in one great circle), then $q$ may also be represented by $\frac{\overline{OC}}{\overline{OB}}$.

In the same way any other versor $r$ may be represented by $\overset{\frown}{DB}$ or $\overset{\frown}{BE}$ and by $\frac{\overline{OB}}{\overline{OD}}$ or $\frac{\overline{OE}}{\overline{OB}}$.

[The line $OB$ in the figure is definite, and is given by the intersection of the planes of the two versors.]

Now $$r\overline{OD}=\overline{OB}, \text{ and } q\overline{OB}=\overline{OC}.$$

Hence $$qr\overline{OD}=\overline{OC},$$

or $qr=\frac{\overline{OC}}{\overline{OD}}$, and may therefore be represented by the arc $\overset{\frown}{DC}$ of a great circle.

But $rq$ is easily seen to be represented by the arc $\overset{\frown}{AE}$.

For $$q\overline{OA}=\overline{OB}, \text{ and } r\overline{OB}=\overline{OE},$$

whence $$rq\overline{OA}=\overline{OE}, \text{ and } rq=\frac{\overline{OE}}{\overline{OA}}.$$

Thus the versors $rq$ and $qr$, though represented by arcs of equal length, are not generally in the same plane and are therefore unequal: unless the planes of $q$ and $r$ coincide.

*Remark.* We see that we have assumed, or defined, in the above proof, that $q\,.\,r\alpha=qr\,.\,\alpha$ and $r\,.\,q\alpha=rq\,.\,\alpha$ in the special case when $q\alpha$, $r\alpha$, $q\,.\,r\alpha$, and $r\,.\,q\alpha$ are all *vectors.*

55. Obviously $\overset{\frown}{CB}$ is $Kq$, $\overset{\frown}{BD}$ is $Kr$, and $\overset{\frown}{CD}$ is $K(qr)$. But $\overset{\frown}{CD}=\overset{\frown}{BD}\,.\,\overset{\frown}{CB}$, as we see by applying both to $OC$. This gives us the very important theorem

$$K(qr)=Kr\,.\,Kq,$$

i.e. *the conjugate of the product of two versors is the product of their* *conjugates in inverted order.* This will, of course, be extended to any number of factors as soon as we have proved the associative property of multiplication. (§ 58 below.)

56. The propositions just proved are, of course, true of quaternions as well as of versors; for the former involve only an additional numerical factor which has reference to the length merely, and not the direction, of a vector (§ 48), and is therefore commutative with all other factors.

57. Seeing thus that the commutative law does not in general hold in the multiplication of quaternions, let us enquire whether the Associative Law holds generally. That is if $p$, $q$, $r$ be three quaternions, have we

$$p \cdot qr = pq \cdot r?$$

This is, of course, obviously true if $p$, $q$, $r$ be numerical quantities, or even any of the imaginaries of algebra. But it cannot be considered as a truism for symbols which do not in general give

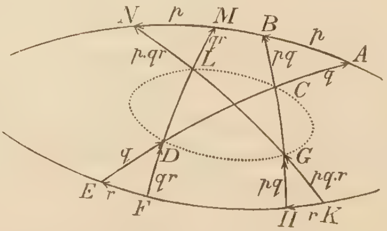
$$pq = qp.$$

We have assumed it, in definition, for the special case when $r$, $qr$, and $pqr$ are all vectors. (§ 54.) But we are not entitled to assume any more than is absolutely required to make our definitions complete.

58. In the first place we remark that $p$, $q$, and $r$ may be considered as versors only, and therefore represented by arcs of great circles on the unit sphere, for their tensors may obviously (§ 48) be divided out from both sides, being commutative with the versors.

Let $\overset{\frown}{AB} = p$, $\overset{\frown}{ED} = \overset{\frown}{CA} = q$, and $\overset{\frown}{FE} = r$.

Join $BC$ and produce the great circle till it meets $EF$ in $H$, and make $\overset{\frown}{KH} = \overset{\frown}{FE} = r$, and $\overset{\frown}{HG} = \overset{\frown}{CB} = pq$ (§ 54).



Join $GK$. Then $\overset{\frown}{KG} = \overset{\frown}{HG} \cdot \overset{\frown}{KH} = pq \cdot r$.

Join $FD$ and produce it to meet $AB$ in $M$. Make

$$\widehat{LM} = \widehat{FD}, \quad \text{and} \quad \widehat{MN} = \widehat{AB},$$

and join $NL$. Then $\widehat{LN} = \widehat{MN} \,.\, \widehat{LM} = p \,.\, qr$.

Hence to show that $\qquad p \,.\, qr = pq \,.\, r$

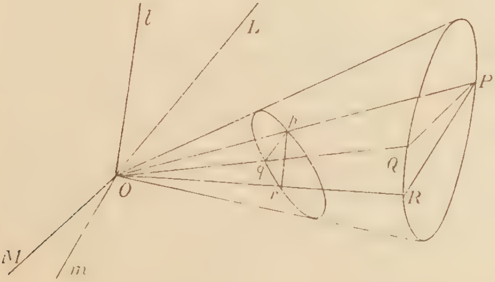
all that is requisite is to prove that $LN$, and $KG$, described as above, are *equal arcs of the same great circle*, since, by the figure, they have evidently similar currency. This is perhaps most easily effected by the help of the fundamental properties of the curves known as *Spherical Conics*. As they are not usually familiar to students, we make a slight digression for the purpose of proving these fundamental properties; after Chasles, by whom and Magnus they were discovered. An independent proof of the associative principle will presently be indicated, and in Chapter VIII. we shall employ quaternions to give an independent proof of the theorems now to be established.

59.* Def. *A spherical conic is the curve of intersection of a cone of the second degree with a sphere, the vertex of the cone being the centre of the sphere.*

Lemma. If a cone have one series of circular sections, it has another series, and any two circles belonging to different series lie on a sphere. This is easily proved as follows.

Describe a sphere, $A$, cutting the cone in one circular section, $C$, and in any other point whatever, and let the side $OpP$ of the cone meet $A$ in $p$, $P$; $P$ being a point in $C$. Then $PO \,.\, Op$ is constant, and, therefore, since $P$ lies in a plane, $p$ lies on a sphere, $a$, passing through $O$. Hence the locus, $c$, of $p$ is a circle, being the intersection of the two spheres $A$ and $a$.

Let $OqQ$ be any other side of the cone, $q$ and $Q$ being points in $c$, $C$ respectively. Then the quadrilateral $qQPp$ is inscribed in a circle (that in which its plane cuts the sphere $A$) and the exterior



angle at $p$ is equal to the interior angle at $Q$. If $OL$, $OM$ be the lines in which the plane $POQ$ cuts the *cyclic planes* (planes through $O$ parallel to the two series of circular sections) they are obviously parallel to $pq$, $QP$, respectively; and therefore

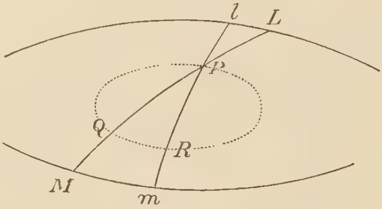
$$\angle LOp = \angle Opq = \angle OQP = \angle MOQ.$$

Let any third side, $OrR$, of the cone be drawn, and let the plane $OPR$ cut the cyclic planes in $Ol$, $Om$ respectively. Then, evidently,

$$\angle lOL = \angle qpr,$$

$$\angle MOm = \angle QPR,$$

and these angles are independent of the position of the points $p$ and $P$, if $Q$ and $R$ be fixed points.



In the annexed section of the above space-diagram by a sphere whose centre is $O$, $lL$, $Mm$ are the great circles which represent the cyclic planes, $PQR$ is the spherical conic which represents the cone. The point $P$ represents the line $OpP$, and so with the others. The propositions above may now be stated thus,

$$\text{Arc } PL = \text{arc } MQ;$$

and, if $Q$ and $R$ be fixed, $Mm$ and $lL$ are constant arcs whatever be the position of $P$.

60. The application to § 58 is now obvious. In the figure of that article we have

$$\frown{FE} = \frown{KH},\quad \frown{ED} = \frown{CA},\quad \frown{HG} = \frown{CB},\quad \frown{LM} = \frown{FD}.$$

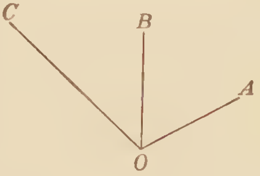
Hence $L$, $C$, $G$, $D$ are points of a spherical conic whose cyclic planes are those of $AB$, $FE$. Hence also $KG$ passes through $L$, and with $LM$ intercepts on $AB$ an arc equal to $\frown{AB}$. That is, it passes through $N$, or $KG$ and $LN$ are arcs of the same great circle: and they are equal, for $G$ and $L$ are points in the spherical conic.

Also, the associative principle holds for any number of quaternion factors. For, obviously,

$$qr \, . \, st = qrs \, . \, t = \&c., \&c.,$$

since we may consider $qr$ as a single quaternion, and the above proof applies directly.

61. That quaternion addition, and therefore also subtraction, is commutative, it is easy to shew.



For if the planes of two quaternions, $q$ and $r$, intersect in the line $OA$, we may take any vector $\overline{OA}$ in that line, and at once find two others, $\overline{OB}$ and $\overline{OC}$, such that

$$\overline{OB} = q\overline{OA},$$

and $$\overline{CO} = r\overline{OA}.$$

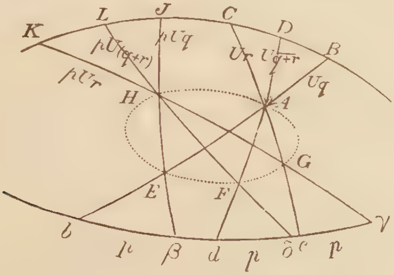
And $$(q + r) \, \overline{OA} = \overline{OB} + \overline{OC} = \overline{OC} + \overline{OB} = (r + q) \, \overline{OA},$$

since *vector* addition is commutative (§ 27).

Here it is obvious that $(q + r) \, \overline{OA}$, being the diagonal of the parallelogram on $\overline{OB}$, $\overline{OC}$, divides the angle between $OB$ and $OC$ in a ratio depending solely on the ratio of the lengths of these lines, i.e. on the ratio of the tensors of $q$ and $r$. This will be useful to us in the proof of the distributive law, to which we proceed.

62. Quaternion multiplication, and therefore division, is distributive. One simple proof of this depends on the possibility, shortly to be proved, of representing *any* quaternion as a linear function of three given rectangular unit-vectors. And when the proposition is thus established, the associative principle may readily be deduced from it.

[But Hamilton seems not to have noticed that we may employ for its proof the properties of Spherical Conics already employed



in demonstrating the truth of the associative principle. For continuity we give an outline of the proof by this process.

Let $\widehat{BA}$, $\widehat{CA}$ represent the versors of $q$ and $r$, and $bc$ the great circle whose plane is that of $p$.

Then, if we take as operand the vector $\overline{OA}$, it is obvious that $U(q+r)$ will be represented by some such arc as $\widehat{DA}$ where $B$, $D$, $C$ are in one great circle; for $(q+r)\,\overline{OA}$ is in the same plane as $q\overline{OA}$ and $r\overline{OA}$, and the relative magnitude of the arcs $BD$ and $DC$ depends solely on the tensors of $q$ and $r$. Produce $BA$, $DA$, $CA$ to meet $bc$ in $b$, $d$, $c$ respectively, and make

$$\widehat{Eb} = \widehat{BA}, \quad \widehat{Fd} = \widehat{DA}, \quad \widehat{Gc} = \widehat{CA}.$$

Also make $\widehat{b\beta} = \widehat{d\delta} = \widehat{c\gamma} = p$. Then $E$, $F$, $G$, $A$ lie on a spherical conic of which $BC$ and $bc$ are the cyclic arcs. And, because $\widehat{b\beta} = \widehat{d\delta} = \widehat{c\gamma}$, $\widehat{\beta E}$, $\widehat{\delta F}$, $\widehat{\gamma G}$, when produced, meet in a point $H$ which is also on the spherical conic (§ 59*). Let these arcs meet $BC$ in $J$, $L$, $K$ respectively. Then we have

$$\widehat{JH} = \widehat{E\beta} = pUq,$$

$$\widehat{LH} = \widehat{F\delta} = p\,U(q+r),$$

$$\widehat{KH} = \widehat{G\gamma} = pUr.$$

Also $$\widehat{LJ} = \widehat{DB},$$

and $$\widehat{KL} = \widehat{CD}.$$

And, on comparing the portions of the figure bounded respectively by $HKJ$ and by $ACB$ we see that (when considered with reference to their effects as factors multiplying $\overline{OH}$ and $\overline{OA}$ respectively)

$pU(q+r)$ bears the same relation to $pUq$ and $pUr$

that $U(q+r)$ bears to $Uq$ and $Ur$.

But $$T(q+r)\,U(q+r) = q+r = Tq\,Uq + Tr\,Ur.$$

Hence $$T(q+r)\,.\,pU(q+r) = Tq\,.\,pUq + Tr\,.\,pUr;$$

or, since the tensors are mere numbers and commutative with all other factors,

$$p(q+r) = pq + pr.$$

In a similar manner it may be proved that

$$(q+r)\,p = qp + rp.$$

And then it follows at once that

$$(p+q)(r+s) = pr + ps + qr + qs,$$

where, by § 61, the order of the partial products is immaterial.]

63. By similar processes to those of § 53 we see that versors, and therefore also quaternions, are subject to the index-law

$$q^m . q^n = q^{m+n},$$

at least so long as $m$ and $n$ are positive integers.

The extension of this property to negative and fractional exponents must be deferred until we have defined a negative or fractional power of a quaternion.

64. We now proceed to the special case of *quadrantal* versors, from whose properties it is easy to deduce all the foregoing results of this chapter. It was, in fact, these properties whose invention by Hamilton in 1843 led almost intuitively to the establishment of the Quaternion Calculus. We shall content ourselves at present with an assumption, which will be shewn to lead to consistent results; but at the end of the chapter we shall shew that no other assumption is possible, following for this purpose a very curious quasi-metaphysical speculation of Hamilton.

65. Suppose we have a system of three mutually perpendicular unit-vectors, drawn from one point, which we may call for shortness $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Suppose also that these are so situated that a positive (i.e. *left-handed*) rotation through a right angle about $\mathbf{i}$ as an axis brings $\mathbf{j}$ to coincide with $\mathbf{k}$. Then it is obvious that positive quadrantal rotation about $\mathbf{j}$ will make $\mathbf{k}$ coincide with $\mathbf{i}$; and, about $\mathbf{k}$, will make $\mathbf{i}$ coincide with $\mathbf{j}$.

For definiteness we may suppose $\mathbf{i}$ to be drawn *eastwards*, $\mathbf{j}$ *northwards*, and $\mathbf{k}$ *upwards*. Then it is obvious that a positive (left-handed) rotation about the eastward line ($\mathbf{i}$) brings the northward line ($\mathbf{j}$) into a vertically upward position ($\mathbf{k}$); and so of the others.

66. Now the operator which turns $\mathbf{j}$ into $\mathbf{k}$ is a quadrantal versor (§ 53); and, as its axis is the vector $\mathbf{i}$, we may call it $i$.

Thus $$\frac{\mathbf{k}}{\mathbf{j}} = i, \quad \text{or} \quad \mathbf{k} = i\mathbf{j} \quad \text{.........................(1).}$$

Similarly we may put $$\frac{\mathbf{i}}{\mathbf{k}} = j, \quad \text{or} \quad \mathbf{i} = j\mathbf{k} \quad \text{.........................(2),}$$

and $$\frac{\mathbf{j}}{\mathbf{i}} = k, \quad \text{or} \quad \mathbf{j} = k\mathbf{i} \quad \text{.........................(3).}$$

[It may be here noticed, merely to shew the symmetry of the system we are explaining, that if the three mutually perpendicular vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be made to revolve about a line equally inclined to all, so that $\mathbf{i}$ is brought to coincide with $\mathbf{j}$, $\mathbf{j}$ will then coincide with $\mathbf{k}$, and $\mathbf{k}$ with $\mathbf{i}$: and the above equations will still hold good, only (1) will become (2), (2) will become (3), and (3) will become (1).]

67. By the results of § 50 we see that

$$\frac{-\mathbf{j}}{\mathbf{k}} = \frac{\mathbf{k}}{\mathbf{j}};$$

i.e. a southward unit-vector bears the same ratio to an upward unit-vector that the latter does to a northward one; and therefore we have

$$\frac{-\mathbf{j}}{\mathbf{k}} = i, \quad \text{or} \quad -\mathbf{j} = i\mathbf{k} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Similarly

$$\frac{-\mathbf{k}}{\mathbf{i}} = j, \quad \text{or} \quad -\mathbf{k} = j\mathbf{i} \ldots\ldots\ldots\ldots\ldots\ldots (5);$$

and

$$\frac{-\mathbf{i}}{\mathbf{j}} = k, \quad \text{or} \quad -\mathbf{i} = k\mathbf{j} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

68. By (4) and (1) we have

$$-\mathbf{j} = i\mathbf{k} = i(i\mathbf{j}) \text{ (by the assumption in § 54)} = i^2\mathbf{j}.$$

Hence $$i^2 = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

And in the same way, (5) and (2) give

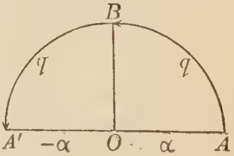
$$j^2 = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and (6) and (3) $$k^2 = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Thus, as the directions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are perfectly arbitrary, we see that *the square of every quadrantal versor is negative unity.*

[Though the following proof is in principle exactly the same as the foregoing, it may perhaps be of use to the student, in shewing him precisely the nature as well as the simplicity of the step we have taken.

Let $ABA'$ be a semicircle, whose centre is $O$, and let $OB$ be perpendicular to $AOA'$.



Then $\frac{\overline{OB}}{\overline{OA}}$, $= q$ suppose, is a quadrantal versor, and is evidently equal to $\frac{\overline{OA'}}{\overline{OB}}$; §§ 50, 53.

Hence $$q^2 = \frac{\overline{OA'}}{\overline{OB}} \cdot \frac{\overline{OB}}{\overline{OA}} = \frac{\overline{OA'}}{\overline{OA}} = -1.]$$

69. Having thus found that the squares of $i$, $j$, $k$ are each equal to negative unity; it only remains that we find the values of their products two and two. For, as we shall see, the result is such as to shew that the value of any other combination whatever of $i$, $j$, $k$ (as factors of a product) may be deduced from the values of these squares and products.

Now it is obvious that

$$\frac{\mathbf{k}}{-\mathbf{i}} = \frac{\mathbf{i}}{\mathbf{k}} = j;$$

(i.e. the versor which turns a westward unit-vector into an upward one will turn the upward into an eastward unit);

or $$\mathbf{k} = j(-\mathbf{i}) = -j\mathbf{i}\text{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Now let us operate on the two equal vectors in (10) by the same versor, $i$, and we have

$$i\mathbf{k} = i(-j\mathbf{i}) = -ij\mathbf{i}.$$

But by (4) and (3)

$$i\mathbf{k} = -\mathbf{j} = -k\mathbf{i}.$$

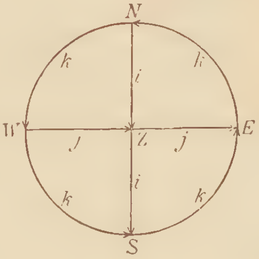
Comparing these equations, we have

$$-ij\mathbf{i} = -k\mathbf{i};$$

or, by § 54 (end), and symmetry gives

$$\left.\begin{aligned} ij &= k, \\ jk &= i, \\ ki &= j. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The meaning of these important equations is very simple; and is, in fact, obvious from our construction in § 54 for the multiplication of versors; as we see by the annexed figure, where we must remember that $i$, $j$, $k$ are quadrantal versors whose planes are at right angles, so that the figure represents a hemisphere divided into quadrantal triangles. [The arrow-heads indicate the direction of each vector arc.]



Thus, to shew that $ij = k$, we have, $O$ being the centre of the sphere, $N$, $E$, $S$, $W$ the north, east, south, and west, and $Z$ the zenith (as in § 65);

$$j\overline{OW} = \overline{OZ},$$

whence $$ij\overline{OW} = i\overline{OZ} = \overline{OS} = k\overline{OW}.$$

---

* The negative sign, being a mere numerical factor, is evidently commutative with $j$; indeed we may, if necessary, easily assure ourselves of the fact that to turn the negative (or reverse) of a vector through a right (or indeed any) angle, is the same thing as to turn the vector through that angle and then reverse it.

70. But, by the same figure,

$$i\overline{ON} = \overline{OZ},$$

whence $$ji\overline{ON} = j\overline{OZ} = \overline{OE} = -\overline{OW} = -k\overline{ON}.$$

71. From this it appears that

and similarly $$\left.\begin{array}{l} ji = -k, \\ kj = -i, \\ ik = -j, \end{array}\right\} \dots\dots\dots\dots (12)$$

and thus, by comparing (11),

$$\left.\begin{array}{l} ij = -ji = k, \\ jk = -kj = i, \\ ki = -ik = j, \end{array}\right\} \quad ((11), (12)).$$

These equations, along with

$$i^2 = j^2 = k^2 = -1 \quad ((7), (8), (9)),$$

contain essentially the whole of Quaternions. But it is easy to see that, for the first group, we may substitute the single equation

$$ijk = -1, \dots\dots\dots\dots (13)$$

since from it, by the help of the values of the squares of $i, j, k$, all the other expressions may be deduced. We may consider it proved in this way, or deduce it afresh from the figure above, thus

$$\begin{aligned} k\overline{ON} &= \overline{OW}, \\ jk\overline{ON} &= j\overline{OW} = \overline{OZ}, \\ ijk\overline{ON} &= ij\overline{OW} = i\overline{OZ} = \overline{OS} = -\overline{ON}. \end{aligned}$$

72. One most important step remains to be made, to wit the assumption referred to in § 64. We have treated $i, j, k$ simply as quadrantal versors; and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as unit-vectors at right angles to each other, and coinciding with the axes of rotation of these versors. But if we collate and compare the equations just proved we have

$$\left\{\begin{array}{ll} i^2 = -1, & \dots\dots\dots\dots (7) \\ \mathbf{i}^2 = -1, & \dots\dots\dots\dots (\S\ 9) \end{array}\right.$$

$$\left\{\begin{array}{ll} ij = \ \ k, & \dots\dots\dots\dots (11) \\ \mathbf{ij} = \ \ \mathbf{k}, & \dots\dots\dots\dots (1) \end{array}\right.$$

$$\left\{\begin{array}{ll} ji = -k, & \dots\dots\dots\dots (12) \\ \mathbf{ji} = -\mathbf{k}, & \dots\dots\dots\dots (5) \end{array}\right.$$

with the other similar groups symmetrically derived from them.

Now the meanings we have assigned to $i$, $j$, $k$ are qu. double pendent of, and not inconsistent with, those assigned to **i**, And it is superfluous to use two sets of characters when one w suffice. Hence it appears that $i$, $j$, $k$ may be substituted for **i**, **j**, **k**, in other words, *a unit-vector when employed as a factor may be considered as a quadrantal versor whose plane is perpendicular to the vector.* (Of course it follows that every vector can be treated as the product of a number and a quadrantal versor.) This is one of the main elements of the singular simplicity of the quaternion calculus.

73. Thus *the product, and therefore the quotient, of two perpendicular vectors is a third vector perpendicular to both.*

Hence the reciprocal (§ 51) of a vector is a vector which has the *opposite* direction to that of the vector, and its length is the reciprocal of the length of the vector.

The conjugate (§ 52) of a vector is simply the vector reversed.

Hence, by § 52, if $\alpha$ be a vector

$$(T\alpha)^2 = \alpha K\alpha = \alpha(-\alpha) = -\alpha^2.$$

74. We may now see that *every versor may be represented by a power of a unit-vector.*

For, if $\alpha$ be any vector perpendicular to $i$ (which is *any* definite unit-vector),

$i\alpha, = \beta$, is a vector equal in length to $\alpha$, but perpendicular to both $i$ and $\alpha$;

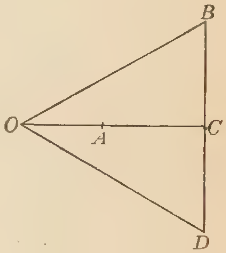
$$i^2\alpha = -\alpha,$$
$$i^3\alpha = -i\alpha = -\beta,$$
$$i^4\alpha = -i\beta = -i^2\alpha = \alpha.$$

Thus, by successive applications of $i$, $\alpha$ is turned round $i$ as an axis through successive right angles. Hence it is natural to *define* $i^m$ *as a versor which turns any vector perpendicular to* $i$ *through* $m$ *right angles in the positive direction of rotation about* $i$ *as an axis.* Here $m$ may have any real value whatever, whole or fractional, for it is easily seen that analogy leads us to interpret a negative value of $m$ as corresponding to rotation in the negative direction.

75. From this again it follows that *any quaternion may be expressed as a power of a vector.* For the tensor and versor elements of the vector may be so chosen that, when raised to the same power, the one may be the tensor and the other the versor of the given quaternion. The vector must be, of course, perpendicular to the plane of the quaternion.

76. And we now see, as an immediate result of the last two ions, that the index-law holds with regard to powers of a aternion (§ 63).

77. So far as we have yet considered it, a quaternion has been regarded as the *product* of a tensor and a versor: we are now to consider it as a *sum*. The easiest method of so analysing it seems to be the following.



Let $\frac{\overline{OB}}{\overline{OA}}$ represent any quaternion. Draw $BC$ perpendicular to $OA$, produced if necessary.

Then, § 19, $\overline{OB} = \overline{OC} + \overline{CB}$.

But, § 22, $\overline{OC} = x\overline{OA}$,

where $x$ is a number, whose sign is the same as that of the cosine of $\angle AOB$.

Also, § 73, since $CB$ is perpendicular to $OA$,

$$\overline{CB} = \gamma\overline{OA},$$

where $\gamma$ is a vector perpendicular to $OA$ and $CB$, i.e. to the plane of the quaternion; and, as the figure is drawn, directed *towards* the reader.

Hence $$\frac{\overline{OB}}{\overline{OA}} = \frac{x\overline{OA} + \gamma\overline{OA}}{\overline{OA}} = x + \gamma.$$

Thus a quaternion, in general, may be decomposed into the sum of two parts, one numerical, the other a vector. Hamilton calls them the SCALAR, and the VECTOR, and denotes them respectively by the letters $S$ and $V$ prefixed to the expression for the quaternion.

78. Hence $q = Sq + Vq$, and if in the above example

$$\frac{\overline{OB}}{\overline{OA}} = q,$$

then $$\overline{OB} = \overline{OC} + \overline{CB} = Sq\,.\,\overline{OA} + Vq\,.\,\overline{OA}\text{*}.$$

The equation above gives

$$\overline{OC} = Sq\,.\,\overline{OA},$$
$$\overline{CB} = Vq\,.\,\overline{OA}.$$

---

* The points are inserted to shew that $S$ and $V$ apply only to $q$, and not to $q\overline{OA}$.

79. If, in the last figure, we produce $BC$ to $D$, so as to double its length, and join $OD$, we have, by § 52,

$$\frac{\overline{OD}}{\overline{OA}} = Kq = SKq + VKq;$$

so that $$\overline{OD} = \overline{OC} + \overline{CD} = SKq \,.\, \overline{OA} + VKq \,.\, \overline{OA}.$$

Hence $$\overline{OC} = SKq \,.\, \overline{OA},$$

and $$\overline{CD} = VKq \,.\, \overline{OA}.$$

Comparing this value of $OC$ with that in last section, we find

$$SKq = Sq, \text{ .................................}(1)$$

or *the scalar of the conjugate of a quaternion is equal to the scalar of the quaternion.*

Again, $CD = -CB$ by the figure, and the substitution of their values gives

$$VKq = -Vq, \text{ ........................ } (2)$$

or *the vector of the conjugate of a quaternion is the vector of the quaternion reversed.*

We may remark that the results of this section are simple consequences of the fact that the symbols $S$, $V$, $K$ are commutative*.

Thus $$SKq = KSq = Sq,$$

since the conjugate of a number is the number itself; and

$$VKq = KVq = -Vq \text{ (§ 73)}.$$

Again, it is obvious that,

$$\Sigma Sq = S\Sigma q, \quad \Sigma Vq = V\Sigma q,$$

and thence $$\Sigma Kq = K\Sigma q.$$

80. Since any vector whatever may be represented by

$$xi + yj + zk$$

where $x$, $y$, $z$ are numbers (or Scalars), and $i$, $j$, $k$ may be any three non-coplanar vectors, §§ 23, 25—though they are usually understood as representing a rectangular system of unit-vectors—and

---

[*] It is curious to compare the properties of these quaternion symbols with those of the Elective Symbols of Logic, as given in BOOLE's wonderful treatise on the *Laws of Thought;* and to think that the same grand science of mathematical analysis, by processes remarkably similar to each other, reveals to us truths in the science of *position* far beyond the powers of the geometer, and truths of deductive reasoning to which unaided thought could never have led the logician.

since any scalar may be denoted by $w$; we may write, for any quaternion $q$, the expression

$$q = w + xi + yj + zk \text{ (§ 78).}$$

Here we have the essential dependence on four distinct numbers, from which the quaternion derives its name, exhibited in the most simple form.

And now we see at once that an equation such as

$$q' = q,$$

where $$q' = w' + x'i + y'j + z'k,$$

involves, of course, the *four* equations

$$w' = w, \quad x' = x, \quad y' = y, \quad z' = z.$$

81. We proceed to indicate another mode of proof of the distributive law of multiplication.

We have already defined, or assumed (§ 61), that

$$\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} = \frac{\beta + \gamma}{\alpha},$$

or $$\beta\alpha^{-1} + \gamma\alpha^{-1} = (\beta + \gamma)\,\alpha^{-1},$$

and have thus been able to understand what is meant by adding two quaternions.

But, writing $\alpha$ for $\alpha^{-1}$, we see that this involves the equality

$$(\beta + \gamma)\,\alpha = \beta\alpha + \gamma\alpha;$$

from which, by taking the conjugates of both sides, we derive

$$\alpha'(\beta' + \gamma') = \alpha'\beta' + \alpha'\gamma' \text{ (§ 55).}$$

And a combination of these results (putting $\beta + \gamma$ for $\alpha'$ in the latter, for instance) gives

$$\begin{aligned}(\beta + \gamma)(\beta' + \gamma') &= (\beta + \gamma)\,\beta' + (\beta + \gamma)\,\gamma' \\ &= \beta\beta' + \gamma\beta' + \beta\gamma' + \gamma\gamma' \text{ by the former.}\end{aligned}$$

Hence *the distributive principle is true in the multiplication of vectors.*

It only remains to shew that it is true as to the scalar and vector parts of a quaternion, and then we shall easily attain the general proof.

Now, if $a$ be any scalar, $\alpha$ any vector, and $q$ any quaternion,

$$(a + \alpha)\,q = aq + \alpha q.$$

For, if $\beta$ be the vector in which the plane of $q$ is intersected by a plane perpendicular to $\alpha$, we can find other two vectors, $\gamma$ and $\delta$, one in each of these planes such that

$$\alpha = \frac{\gamma}{\beta}, \qquad q = \frac{\beta}{\delta}.$$

And, of course, $a$ may be written $\frac{a\beta}{\beta}$; so that

$$\begin{aligned}(a + \alpha)\, q &= \frac{a\beta + \gamma}{\beta} \cdot \frac{\beta}{\delta} = \frac{a\beta + \gamma}{\delta} \\ &= a\frac{\beta}{\delta} + \frac{\gamma}{\delta} = a\frac{\beta}{\delta} + \frac{\gamma}{\beta} \cdot \frac{\beta}{\delta} \\ &= aq + \alpha q.\end{aligned}$$

And the conjugate may be written

$$q'\,(a' + \alpha') = q'a' + q'\alpha' \text{ (§ 55).}$$

Hence, generally,

$$(a + \alpha)\,(b + \beta) = ab + a\beta + b\alpha + \alpha\beta\,;$$

or, breaking up $a$ and $b$ each into the sum of two scalars, and $\alpha$, $\beta$ each into the sum of two vectors,

$$\begin{aligned}&(a_1 + a_2 + \alpha_1 + \alpha_2)\,(b_1 + b_2 + \beta_1 + \beta_2) \\ &\quad = (a_1 + a_2)\,(b_1 + b_2) + (a_1 + a_2)\,(\beta_1 + \beta_2) + (b_1 + b_2)\,(\alpha_1 + \alpha_2) \\ &\qquad + (\alpha_1 + \alpha_2)\,(\beta_1 + \beta_2)\end{aligned}$$

(by what precedes, all the factors on the right are distributive, so that we may easily put it in the form)

$$\begin{aligned}&= (a_1 + \alpha_1)\,(b_1 + \beta_1) + (a_1 + \alpha_1)\,(b_2 + \beta_2) + (a_2 + \alpha_2)\,(b_1 + \beta_1) \\ &\qquad + (a_2 + \alpha_2)\,(b_2 + \beta_2).\end{aligned}$$

Putting $a_1 + \alpha_1 = p, \quad a_2 + \alpha_2 = q, \quad b_1 + \beta_1 = r, \quad b_2 + \beta_2 = s,$

we have $$(p + q)\,(r + s) = pr + ps + qr + qs.$$

82. Cayley suggests that the laws of quaternion multiplication may be derived more directly from those of vector multiplication, supposed to be already established. Thus, let $\alpha$ be the unit vector perpendicular to the vector parts of $q$ and of $q'$. Then let

$$\rho = q \,.\, \alpha, \quad \sigma = -\alpha \,.\, q',$$

as is evidently permissible, and we have

$$\rho\alpha = q \,.\, \alpha\alpha = -q\,; \quad \alpha\sigma = -\alpha\alpha \,.\, q' = q'.$$

so that $$-q \,.\, q' = \rho\alpha \,.\, \alpha\sigma = -\rho \,.\, \sigma.$$

The student may easily extend this process.

For variety, we shall now for a time forsake the geometrical mode of proof we have hitherto adopted, and deduce some of our next steps from the analytical expression for a quaternion given in § 80, and the properties of a rectangular system of unit-vectors as in § 71.

We will commence by proving the result of § 77 anew.

83. Let
$$\alpha = xi + yj + zk,$$
$$\beta = x'i + y'j + z'k.$$

Then, because by § 71 every product or quotient of $i, j, k$ is reducible to one of them or to a number, we are entitled to assume

$$q = \frac{\beta}{\alpha} = \omega + \xi i + \eta j + \zeta k,$$

where $\omega$, $\xi$, $\eta$, $\zeta$ are numbers. This is the proposition of § 80.

[Of course, with this expression for a quaternion, there is no necessity for a formal proof of such equations as

$$p + (q + r) = (p + q) + r,$$

where the various sums are to be interpreted as in § 61.

All such things become obvious in view of the properties of $i, j, k$.]

84. But it may be interesting to find $\omega$, $\xi$, $\eta$, $\zeta$ in terms of $x, y, z, x', y', z'$.

We have
$$\beta = q\alpha,$$
or
$$x'i + y'j + z'k = (\omega + \xi i + \eta j + \zeta k)(xi + yj + zk)$$
$$= -(\xi x + \eta y + \zeta z) + (\omega x + \eta z - \zeta y)i + (\omega y + \zeta x - \xi z)j + (\omega z + \xi y - \eta x)k,$$
as we easily see by the expressions for the powers and products of $i, j, k$, given in § 71. But the student must pay particular attention to the *order* of the factors, else he is certain to make mistakes.

This (§ 80) resolves itself into the four equations

$$\begin{aligned} 0 &= \quad\;\; \xi x + \eta y + \zeta z, \\ x' &= \omega x \qquad\; + \eta z - \zeta y, \\ y' &= \omega y - \xi z \qquad\; + \zeta x, \\ z' &= \omega z + \xi y - \eta x. \end{aligned}$$

The three last equations give

$$xx' + yy' + zz' = \omega(x^2 + y^2 + z^2),$$

which determines $\omega$.

Also we have, from the same three, by the help of the first,

$$\xi x' + \eta y' + \zeta z' = 0;$$

which, combined with the first, gives

$$\frac{\xi}{yz'-zy'}=\frac{\eta}{zx'-xz'}=\frac{\zeta}{xy'-yx'};$$

and the common value of these three fractions is then easily seen to be

$$\frac{1}{x^2+y^2+z^2}.$$

It is easy enough to interpret these expressions by means of ordinary cöordinate geometry: but a much simpler process will be furnished by quaternions themselves in the next chapter, and, in giving it, we shall refer back to this section.

85. The associative law of multiplication is now to be proved by means of the distributive (§ 81). We leave the proof to the student. He has merely to multiply together the factors

$$w+xi+yj+zk,\quad w'+x'i+y'j+z'k,\quad \text{and}\quad w''+x''i+y''j+z''k,$$

as follows :—

First, multiply the third factor by the second, and then multiply the product by the first; next, multiply the second factor by the first and employ the product to multiply the third: always remembering that the multiplier in any product is placed *before* the multiplicand. He will find the scalar parts and the coefficients of $i, j, k$, in these products, respectively equal, each to each.

86. With the same expressions for $\alpha$, $\beta$, as in section 83, we have

$$\begin{aligned}\alpha\beta &= (xi+yj+zk)\,(x'i+y'j+z'k)\\ &= -(xx'+yy'+zz')+(yz'-zy')\,i+(zx'-xz')\,j+(xy'-yx')\,k.\end{aligned}$$

But we have also

$$\beta\alpha = -(xx'+yy'+zz')-(yz'-zy')\,i-(zx'-xz')\,j-(xy'-yx')\,k.$$

The only difference is in the *sign* of the vector parts.

Hence $$S\alpha\beta = S\beta\alpha, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (1)$$

$$V\alpha\beta = -V\beta\alpha, \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (2)$$

also $$\alpha\beta+\beta\alpha = 2S\alpha\beta, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (3)$$

$$\alpha\beta-\beta\alpha = 2V\alpha\beta, \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (4)$$

and, finally, by § 79, $$\alpha\beta = K.\beta\alpha. \ldots\ldots\ldots\ldots\ldots\ldots\quad (5).$$

87. If $\alpha=\beta$ we have of course (§ 25)

$$x=x',\qquad y=y',\qquad z=z'.$$

and the formulae of last section become

$$\alpha\beta = \beta\alpha = \alpha^2 = -(x^2 + y^2 + z^2);$$

which was anticipated in § 73, where we proved the formula

$$(T\alpha)^2 = -\alpha^2,$$

and also, to a certain extent, in § 25.

88. Now let $q$ and $r$ be any quaternions, then

$$\begin{aligned} S.qr &= S.(Sq + Vq)(Sr + Vr), \\ &= S.(Sq\,Sr + Sr.Vq + Sq.Vr + VqVr), \\ &= Sq\,Sr + S.VqVr, \end{aligned}$$

since the two middle terms are vectors.

Similarly, $$S.rq = SrSq + S.VrVq.$$

Hence, since by (1) of § 86 we have

$$S.VqVr = S.VrVq,$$

we see that $$S.qr = S.rq, \quad\text{.......................}\quad (1)$$

a formula of considerable importance.

It may easily be extended to any number of quaternions, because, $r$ being arbitrary, we may put for it $rs$. Thus we have

$$\begin{aligned} S.qrs &= S.rsq, \\ &= S.sqr \end{aligned}$$

by a second application of the process. In words, we have the theorem—*the scalar of the product of any number of given quaternions depends only upon the cyclical order in which they are arranged.*

89. An important case is that of three factors, each a vector. The formula then becomes

$$S.\alpha\beta\gamma = S.\beta\gamma\alpha = S.\gamma\alpha\beta.$$

But $$\begin{aligned} S.\alpha\beta\gamma &= S\alpha(S\beta\gamma + V\beta\gamma) \\ &= S\alpha V\beta\gamma, && \text{since } \alpha S\beta\gamma \text{ is a vector,} \\ &= -S\alpha V\gamma\beta, && \text{by (2) of § 86,} \\ &= -S\alpha(S\gamma\beta + V\gamma\beta) \\ &= -S.\alpha\gamma\beta. \end{aligned}$$

Hence *the scalar of the product of three vectors changes sign when the cyclical order is altered.*

By the results of §§ 55, 73, 79 we see that, for any number of vectors, we have

$$K.\alpha\beta\gamma\ldots\phi\chi = \pm\chi\phi\ldots\gamma\beta\alpha$$

(the positive sign belonging to the product of an even number of vectors) so that

$$S.\alpha\beta\ldots\phi\chi = \pm S.\chi\phi\ldots\beta\alpha.$$

Similarly

$$V.\alpha\beta\ldots\phi\chi = \mp V.\chi\phi\ldots\beta\alpha.$$

Thus we may generalize (3) and (4) of § 86 into

$$2S.\alpha\beta\ldots\phi\chi = \alpha\beta\ldots\phi\chi \pm \chi\phi\ldots\beta\alpha,$$

$$2V.\alpha\beta\ldots\phi\chi = \alpha\beta\ldots\phi\chi \mp \chi\phi\ldots\beta\alpha,$$

the upper sign still being used when the number of factors is even.

Other curious propositions connected with this will be given later (some, indeed, will be found in the Examples appended to this chapter), as we wish to develop the really fundamental formulae in as compact a form as possible.

90. By (4) of § 86,

$$2V\beta\gamma = \beta\gamma - \gamma\beta.$$

Hence $\quad 2V.\alpha V\beta\gamma = V.\alpha(\beta\gamma - \gamma\beta)$

(by multiplying both by $\alpha$, and taking the vector parts of each side)

$$= V(\alpha\beta\gamma + \beta\alpha\gamma - \beta\alpha\gamma - \alpha\gamma\beta)$$

(by introducing the null term $\beta\alpha\gamma - \beta\alpha\gamma$).

That is

$$2V.\alpha V\beta\gamma = V.(\alpha\beta + \beta\alpha)\gamma - V(\beta S\alpha\gamma + \beta V\alpha\gamma + S\alpha\gamma.\beta + V\alpha\gamma.\beta)$$

$$= V.(2S\alpha\beta)\gamma - 2V\beta S\alpha\gamma$$

(if we notice that $V(V\alpha\gamma.\beta) = -V.\beta V\alpha\gamma$, by (2) of § 86).

Hence $\quad V.\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\gamma\alpha \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$

a formula of constant occurrence.

Adding $\alpha S\beta\gamma$ to both sides, we get another most valuable formula

$$V.\alpha\beta\gamma = \alpha S\beta\gamma - \beta S\gamma\alpha + \gamma S\alpha\beta \;\ldots\ldots\ldots\ldots\; (2);$$

and the form of this shews that we may interchange $\gamma$ and $\alpha$ without altering the right-hand member. This gives

$$V.\alpha\beta\gamma = V.\gamma\beta\alpha,$$

a formula which may be greatly extended. (See § 89, above.)

Another simple mode of establishing (2) is as follows:—

$$K\,.\,\alpha\beta\gamma=-\gamma\beta\alpha,$$

$$\begin{aligned}\therefore 2V\,.\,\alpha\beta\gamma &= \alpha\beta\gamma - K\,.\,\alpha\beta\gamma \text{ (by § 79 (2))}\\ &= \alpha\beta\gamma+\gamma\beta\alpha\\ &= \alpha(\beta\gamma+\gamma\beta)-(\alpha\gamma+\gamma\alpha)\beta+\gamma(\alpha\beta+\beta\alpha)\\ &= 2\alpha S\beta\gamma-2\beta S\alpha\gamma+2\gamma S\alpha\beta.\end{aligned}$$

91. We have also

$$\begin{aligned}V V\alpha\beta V\gamma\delta &= -VV\gamma\delta V\alpha\beta \quad \text{by (2) of § 86:}\\ &= \delta S\gamma V\alpha\beta-\gamma S\delta V\alpha\beta=\delta S\,.\,\alpha\beta\gamma-\gamma S\,.\,\alpha\beta\delta\\ &= -\beta S\alpha V\gamma\delta+\alpha S\beta V\gamma\delta=-\beta S\,.\,\alpha\gamma\delta+\alpha S\,.\,\beta\gamma\delta,\end{aligned}$$

all of these being arrived at by the help of § 90 (1) and of § 89; and by treating alternately $V\alpha\beta$ and $V\gamma\delta$ as *simple* vectors.

Equating two of these values, we have

$$\delta S\,.\,\alpha\beta\gamma=\alpha S\,.\,\beta\gamma\delta+\beta S\,.\,\gamma\alpha\delta+\gamma S\,.\,\alpha\beta\delta \quad\ldots\ldots\ldots (3),$$

a very useful formula, expressing any vector whatever in terms of three given vectors. [This, of course, presupposes that $\alpha$, $\beta$, $\gamma$ are not coplanar, § 23. In fact, if they be coplanar, the factor $S\,.\,\alpha\beta\gamma$ vanishes, and thus (3) does not give an expression for $\delta$. This will be shewn in § 101 below.]

92. That such an expression as (3) is possible we knew already by § 23. For variety we may seek another expression of a similar character, by a process which differs entirely from that employed in last section.

$\alpha$, $\beta$, $\gamma$ being any three non-coplanar vectors, we may derive from them three others $V\alpha\beta$, $V\beta\gamma$, $V\gamma\alpha$; and, as these will not be coplanar, any other vector $\delta$ may be expressed as the sum of the three, each multiplied by some scalar. It is required to find this expression for $\delta$.

Let $\qquad \delta=xV\alpha\beta+yV\beta\gamma+zV\gamma\alpha.$

Then $\qquad S\gamma\delta=xS\,.\,\gamma\alpha\beta=xS\,.\,\alpha\beta\gamma,$

the terms in $y$ and $z$ going out, because

$$S\gamma V\beta\gamma=S\,.\,\gamma\beta\gamma=S\beta\gamma^2=\gamma^2 S\beta=0,$$

for $\gamma^2$ is (§ 73) a number.

Similarly $\qquad S\beta\delta=zS\,.\,\beta\gamma\alpha=zS\,.\,\alpha\beta\gamma,$

and $\qquad S\alpha\delta=yS\,.\,\alpha\beta\gamma.$

Thus $\qquad \delta S\,.\,\alpha\beta\gamma=V\alpha\beta S\gamma\delta+V\beta\gamma S\alpha\delta+V\gamma\alpha S\beta\delta \quad\ldots\ldots\ldots (4).$

93. We conclude the chapter by shewing (as promised in § 64) that the assumption that the product of two parallel vectors is a number, and the product of two perpendicular vectors a third vector perpendicular to both, is not only useful and convenient, but absolutely inevitable, if our system is to deal indifferently with all directions in space. We abridge Hamilton's reasoning.

Suppose that there is no direction in space pre-eminent, and that the product of two vectors is something which has quantity, so as to vary in amount if the factors are changed, and to have its sign changed if that of one of them is reversed; if the vectors be parallel, their product cannot be, in whole or in part, a vector *inclined* to them, for there is nothing to determine the direction in which it must lie. It cannot be a vector *parallel* to them; for by changing the signs of both factors the product is unchanged, whereas, as the whole system has been reversed, the product vector ought to have been reversed. Hence it must be a number. Again, the product of two perpendicular vectors cannot be wholly or partly a number, because on inverting one of them the sign of that number ought to change; but inverting one of them is simply equivalent to a rotation through two right angles about the other, and (from the symmetry of space) ought to leave the number unchanged. Hence the product of two perpendicular vectors must be a vector, and a simple extension of the same reasoning shews that it must be perpendicular to each of the factors. It is easy to carry this farther, but enough has been said to shew the character of the reasoning.

## EXAMPLES TO CHAPTER II.

1. It is obvious from the properties of polar triangles that any mode of representing versors by the *sides* of a spherical triangle must have an equivalent statement in which they are represented by *angles* in the polar triangle.

Shew directly that the product of two versors represented by two angles of a spherical triangle is a third versor represented by the *supplement* of the remaining angle of the triangle; and determine the rule which connects the *directions* in which these angles are to be measured.

2. Hence derive another proof that we have not generally

$$pq = qp.$$

3. Hence shew that the proof of the associative principle, § 57, may be made to depend upon the fact that if from any point of the sphere tangent arcs be drawn to a spherical conic, and also arcs to the foci, the inclination of either tangent arc to one of the focal arcs is equal to that of the other tangent arc to the other focal arc.

4. Prove the formulae

$$2S \,.\, \alpha\beta\gamma = \alpha\beta\gamma - \gamma\beta\alpha,$$
$$2V \,.\, \alpha\beta\gamma = \alpha\beta\gamma + \gamma\beta\alpha.$$

5. Shew that, whatever odd number of vectors be represented by $\alpha$, $\beta$, $\gamma$, &c., we have always

$$V \,.\, \alpha\beta\gamma\delta\epsilon = V \,.\, \epsilon\delta\gamma\beta\alpha,$$
$$V \,.\, \alpha\beta\gamma\delta\epsilon\zeta\eta = V \,.\, \eta\zeta\epsilon\delta\gamma\beta\alpha, \text{ \&c.}$$

6. Shew that

$$S \,.\, V\alpha\beta V\beta\gamma V\gamma\alpha = -(S \,.\, \alpha\beta\gamma)^2,$$
$$V \,.\, V\alpha\beta V\beta\gamma V\gamma\alpha = V\alpha\beta\,(\gamma^2 S\alpha\beta - S\beta\gamma S\gamma\alpha) + \ldots\ldots,$$

and $V(V\alpha\beta V \,.\, V\beta\gamma V\gamma\alpha) = (\beta S\alpha\gamma - \alpha S\beta\gamma)\, S \,.\, \alpha\beta\gamma.$

7. If $\alpha, \beta, \gamma$ be any vectors at right angles to each other, shew that

$$(\alpha^3 + \beta^3 + \gamma^3)\, S \,.\, \alpha\beta\gamma = \alpha^4 V\beta\gamma + \beta^4 V\gamma\alpha + \gamma^4 V\alpha\beta.$$
$$(\alpha^{2n-1} + \beta^{2n-1} + \gamma^{2n-1})\, S \,.\, \alpha\beta\gamma = \alpha^{2n} V\beta\gamma + \beta^{2n} V\gamma\alpha + \gamma^{2n} V\alpha\beta.$$

8. If $\alpha$, $\beta$, $\gamma$ be non-coplanar vectors, find the relations among the six scalars, $x$, $y$, $z$ and $\xi$, $\eta$, $\zeta$, which are implied in the equation $x\alpha + y\beta + z\gamma = \xi V\beta\gamma + \eta V\gamma\alpha + \zeta V\alpha\beta$.

9. If $\alpha$, $\beta$, $\gamma$ be any three non-coplanar vectors, express any fourth vector, $\delta$, as a linear function of each of the following sets of three derived vectors,

$$V \,.\, \gamma\alpha\beta, \quad V \,.\, \alpha\beta\gamma, \quad V \,.\, \beta\gamma\alpha,$$

and $V \,.\, V\alpha\beta V\beta\gamma V\gamma\alpha, \quad V \,.\, V\beta\gamma V\gamma\alpha V\alpha\beta, \quad V \,.\, V\gamma\alpha V\alpha\beta V\beta\gamma.$

10. Eliminate $\rho$ from the equations

$$S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c, \quad S\delta\rho = d,$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are vectors, and $a$, $b$, $c$, $d$ scalars.

11. In any quadrilateral, plane or gauche, the sum of the squares of the diagonals is double the sum of the squares of the lines joining the middle points of opposite sides.

# CHAPTER III.

## INTERPRETATIONS AND TRANSFORMATIONS OF QUATERNION EXPRESSIONS.

94. AMONG the most useful characteristics of the Calculus of Quaternions, the ease of interpreting its formulae geometrically, and the extraordinary variety of transformations of which the simplest expressions are susceptible, deserve a prominent place. We devote this Chapter to some of the more simple of these, together with a few of somewhat more complex character but of constant occurrence in geometrical and physical investigations. Others will appear in every succeeding Chapter. It is here, perhaps, that the student is likely to feel most strongly the peculiar difficulties of the new Calculus. But on that very account he should endeavour to master them, for the variety of forms which any one formula may assume, though puzzling to the beginner, is of the utmost advantage to the advanced student, not alone as aiding him in the solution of complex questions, but as affording an invaluable mental discipline.

95. If we refer again to the figure of § 77 we see that

$$OC = OB \cos AOB,$$

$$CB = OB \sin AOB.$$

Hence, if $\overline{OA} = \alpha$, $\overline{OB} = \beta$, and $\angle AOB = \theta$, we have

$$OB = T\beta, \qquad OA = T\alpha,$$

$$OC = T\beta \cos\theta, \qquad CB = T\beta \sin\theta.$$

Hence
$$S\frac{\beta}{\alpha} = \frac{OC}{OA} = \frac{T\beta}{T\alpha}\cos\theta.$$

Similarly
$$TV\frac{\beta}{\alpha} = \frac{CB}{OA} = \frac{T\beta}{T\alpha}\sin\theta.$$

Hence, if $\eta$ be a unit-vector perpendicular to $\alpha$ and $\beta$, and such that positive rotation about it, through the angle $\theta$, turns $\alpha$ towards $\beta$, or

$$\eta = \frac{U\overline{CB}}{U\overline{OA}} = U\frac{\overline{CB}}{\overline{OA}} = UV\frac{\beta}{\alpha},$$

we have $$V\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}\sin\theta \,.\, \eta. \quad \text{(See, again, § 84.)}$$

96. In the same way, or by putting

$$\begin{aligned} \alpha\beta &= S\alpha\beta + V\alpha\beta, \\ &= S\beta\alpha - V\beta\alpha, \\ &= \alpha^2\left(S\frac{\beta}{\alpha} - V\frac{\beta}{\alpha}\right), \\ &= T\alpha^2 \,.\, \left(-S\frac{\beta}{\alpha} + V\frac{\beta}{\alpha}\right), \end{aligned}$$

we may shew that

$$\begin{aligned} S\alpha\beta &= -\,T\alpha\, T\beta\cos\theta, \\ TV\alpha\beta &= \quad T\alpha\, T\beta\sin\theta, \end{aligned}$$

and $$V\alpha\beta = \quad T\alpha\, T\beta\sin\theta \,.\, \eta$$

where $$\eta = UV\alpha\beta = U(-V\beta\alpha) = UV\frac{\beta}{\alpha}.$$

Thus *the scalar of the product of two vectors is the continued product of their tensors and of the cosine of the supplement of the contained angle.*

*The tensor of the vector of the product of two vectors is the continued product of their tensors and the sine of the contained angle; and the versor of the same is a unit-vector perpendicular to both, and such that the rotation about it from the first vector* (i. e. *the multiplier*) *to the second is left-handed or positive.*

*Hence also $TV\alpha\beta$ is double the area of the triangle two of whose sides are $\alpha$, $\beta$.*

97. (*a*) In any plane triangle $ABC$ we have

$$\overline{AC} = \overline{AB} + \overline{BC}.$$

Hence $$\overline{AC}^2 = S\,.\,\overline{AC}\,\overline{AC} = S\,.\,\overline{AC}(\overline{AB} + \overline{BC}).$$

With the usual notation for a plane triangle the interpretation of this formula is

$$-b^2 = -bc\cos A - ab\cos C,$$

or $$b = \quad a\cos C + c\cos A.$$

(*b*) Again we have, obviously,

$$V.\overline{AB}\,\overline{AC} = V.\overline{AB}\,(\overline{AB} + \overline{BC})$$

$$= V.\overline{AB}\,\overline{BC},$$

or $$cb \sin A = ca \sin B,$$

whence $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

These are truths, but not truisms, as we might have been led to fancy from the excessive simplicity of the process employed.

98. From § 96 it follows that, if $\alpha$ and $\beta$ be both actual (i.e. real and non-evanescent) vectors, the equation

$$S\alpha\beta = 0$$

shews that $\cos\theta = 0$, or that $\alpha$ *is perpendicular to* $\beta$. And, in fact, we know already that the product of two perpendicular vectors is a vector.

Again, if $$V\alpha\beta = 0,$$

we must have $\sin\theta = 0$, or $\alpha$ *is parallel to* $\beta$. We know already that the product of two parallel vectors is a scalar.

Hence we see that

$$S\alpha\beta = 0$$

is equivalent to $$\alpha = V\gamma\beta,$$

where $\gamma$ is an undetermined vector; and that

$$V\alpha\beta = 0$$

is equivalent to $$\alpha = x\beta,$$

where $x$ is an undetermined scalar.

99. If we write, as in §§ 83, 84,

$$\alpha = ix + jy + kz,$$

$$\beta = ix' + jy' + kz',$$

we have, at once, by § 86,

$$S\alpha\beta = -xx' - yy' - zz'$$

$$= -rr'\left(\frac{x}{r}\frac{x'}{r'} + \frac{y}{r}\frac{y'}{r'} + \frac{z}{r}\frac{z'}{r'}\right)$$

where $$r = \sqrt{x^2 + y^2 + z^2}, \quad r' = \sqrt{x'^2 + y'^2 + z'^2}.$$

Also $$V\alpha\beta = rr'\left\{\frac{yz' - zy'}{rr'}\,i + \frac{zx' - xz'}{rr'}\,j + \frac{xy' - yx'}{rr'}\,k\right\}.$$

These express in Cartesian cöordinates the propositions we have just proved. In commencing the subject it may perhaps assist the student to see these more familiar forms for the quaternion expressions; and he will doubtless be induced by their appearance to prosecute the subject, since he cannot fail even at this stage to see how much more simple the quaternion expressions are than those to which he has been accustomed.

100. The expression $$S \,.\, \alpha\beta\gamma$$
may be written $$SV(\alpha\beta)\,\gamma,$$
because the quaternion $\alpha\beta\gamma$ may be broken up into
$$S(\alpha\beta)\,\gamma + V(\alpha\beta)\,\gamma$$
of which the first term is a vector.

But, by § 96,
$$SV(\alpha\beta)\,\gamma = T\alpha\, T\beta \sin\theta\, S\eta\gamma.$$
Here $T\eta = 1$, let $\phi$ be the angle between $\eta$ and $\gamma$, then finally
$$S \,.\, \alpha\beta\gamma = -\, T\alpha\, T\beta\, T\gamma \sin\theta \cos\phi.$$
But as $\eta$ is perpendicular to $\alpha$ and $\beta$, $T\gamma\cos\phi$ is the length of the perpendicular from the extremity of $\gamma$ upon the plane of $\alpha, \beta$. And as the product of the other three factors is (§ 96) the area of the parallelogram two of whose sides are $\alpha, \beta$, we see that the magnitude of $S \,.\, \alpha\beta\gamma$, independent of its sign, is *the volume of the parallelepiped of which three cöordinate edges are* $\alpha, \beta, \gamma$; or six times the volume of the pyramid which has $\alpha, \beta, \gamma$ for edges.

101. Hence the equation
$$S \,.\, \alpha\beta\gamma = 0,$$
if we suppose $\alpha, \beta, \gamma$ to be actual vectors, shews either that
$$\sin\theta = 0,$$
or $$\cos\phi = 0,$$
i. e. *two of the three vectors are parallel*, or *all three are parallel to one plane.*

This is consistent with previous results, for if $\gamma = p\beta$ we have
$$S \,.\, \alpha\beta\gamma = pS \,.\, \alpha\beta^2 = 0;$$
and, if $\gamma$ be coplanar with $\alpha, \beta$, we have $\gamma = p\alpha + q\beta$, and
$$S \,.\, \alpha\beta\gamma = S \,.\, \alpha\beta\,(p\alpha + q\beta) = 0.$$

102. This property of the expression $S.\alpha\beta\gamma$ prepares us to find that it is a determinant. And, in fact, if we take $\alpha$, $\beta$ as in § 83, and in addition $\gamma = ix'' + jy'' + kz''$,
we have at once

$$S.\alpha\beta\gamma = -x''(yz' - zy') - y''(zx' - xz') - z''(xy' - yx'),$$
$$= -\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}.$$

The determinant changes sign if we make any two rows change places. This is the proposition we met with before (§ 89) in the form
$$S.\alpha\beta\gamma = -S.\beta\alpha\gamma = S.\beta\gamma\alpha, \text{ \&c.}$$

If we take three new vectors

$$\alpha_1 = ix + jx' + kx'',$$
$$\beta_1 = iy + jy' + ky'',$$
$$\gamma_1 = iz + jz' + kz'',$$

we thus see that they are coplanar if $\alpha$, $\beta$, $\gamma$ are so. That is, if
$$S.\alpha\beta\gamma = 0,$$
then
$$S.\alpha_1\beta_1\gamma_1 = 0.$$

103. We have, by § 52,

$$(Tq)^2 = qKq = (Sq + Vq)(Sq - Vq) \quad (\S\ 79),$$
$$= (Sq)^3 - (Vq)^2 \text{ by algebra,}$$
$$= (Sq)^2 + (TVq)^2 \ (\S\ 73).$$

If $q = \alpha\beta$, we have $Kq = \beta\alpha$, and the formula becomes
$$\alpha\beta\,.\,\beta\alpha = \alpha^2\beta^2 = (S\alpha\beta)^2 - (V\alpha\beta)^2.$$

In Cartesian cöordinates this is

$$(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2)$$
$$= (xx' + yy' + zz')^2 + (yz' - zy')^2 + (zx' - xz')^2 + (xy' - yx')^2.$$

More generally we have

$$(T(qr))^2 = (Tq)^2 (Tr)^2$$
$$= (S.qr)^2 - (V.qr)^2.$$

If we write
$$q = w + \alpha = w + ix + jy + kz,$$
$$r = w' + \beta = w' + ix' + jy' + kz';$$
this becomes

$$(w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2)$$
$$= (ww' - xx' - yy' - zz')^2 + (wx' + w'x + yz' - zy')^2$$
$$+ (wy' + w'y + zx' - xz')^2 + (wz' + w'z + xy' - yx')^2,$$

a formula of algebra due to Euler.

104. We have, of course, by multiplication,

$$(\alpha+\beta)^2=\alpha^2+\alpha\beta+\beta\alpha+\beta^2=\alpha^2+2S\alpha\beta+\beta^2 \text{ (§ 86 (3)).}$$

Translating into the usual notation of plane trigonometry, this becomes $$c^2=a^2-2ab\cos C+b^2,$$

the common formula.

Again, $V.(\alpha+\beta)(\alpha-\beta)=-V\alpha\beta+V\beta\alpha=-2V\alpha\beta$ (§ 86 (2)). Taking tensors of both sides we have the theorem, *the parallelogram whose sides are parallel and equal to the diagonals of a given parallelogram, has double its area* (§ 96).

Also $$S(\alpha+\beta)(\alpha-\beta)=\alpha^2-\beta^2,$$

and vanishes only when $\alpha^2=\beta^2$, or $T\alpha=T\beta$; that is, *the diagonals of a parallelogram are at right angles to one another, when, and only when, it is a rhombus.*

Later it will be shewn that this contains a proof that the angle in a semicircle is a right angle.

105. The expression $$\rho=\alpha\beta\alpha^{-1}$$

obviously denotes a vector whose tensor is equal to that of $\beta$.

But we have $$S.\beta\alpha\rho=0,$$

so that $\rho$ is in the plane of $\alpha$, $\beta$.

Also we have $$S\alpha\rho=S\alpha\beta,$$

so that $\beta$ and $\rho$ make equal angles with $\alpha$, evidently on opposite sides of it. Thus if $\alpha$ be the perpendicular to a reflecting surface and $\beta$ the path of an incident ray, $-\rho$ will be the path of the reflected ray.

Another mode of obtaining these results is to expand the above expression, thus, § 90 (2),

$$\begin{aligned}\rho&=2\alpha^{-1}S\alpha\beta-\beta\\&=2\alpha^{-1}S\alpha\beta-\alpha^{-1}(S\alpha\beta+V\alpha\beta)\\&=\alpha^{-1}(S\alpha\beta-V\alpha\beta),\end{aligned}$$

so that in the figure of § 77 we see that if $\overline{OA}=\alpha$, and $\overline{OB}=\beta$, we have $\overline{OD}=\rho=\alpha\beta\alpha^{-1}$.

Or, again, we may get the result at once by transforming the equation to $\frac{\rho}{\alpha}=K(\alpha^{-1}\rho)=K\frac{\beta}{\alpha}$.

106. For any three coplanar vectors the expression

$$\rho = \alpha\beta\gamma$$

is (§ 101) a vector. It is interesting to determine what this vector is. The reader will easily see that if a circle be described about the triangle, two of whose sides are (in order) $\alpha$ and $\beta$, and if from the extremity of $\beta$ a line parallel to $\gamma$ be drawn, again cutting the circle, the vector joining the point of intersection with the origin of $\alpha$ is the direction of the vector $\alpha\beta\gamma$. For we may write it in the form

$$\rho = \alpha\beta^2\beta^{-1}\gamma = -(T\beta)^2\alpha\beta^{-1}\gamma = -(T\beta)^2\frac{\alpha}{\beta}\gamma,$$

which shews that the *versor* $\left(\frac{\alpha}{\beta}\right)$ which turns $\beta$ into a direction parallel to $\alpha$, turns $\gamma$ into a direction parallel to $\rho$. And this expresses the long-known property of opposite angles of a quadrilateral inscribed in a circle.

Hence if $\alpha$, $\beta$, $\gamma$ be the sides of a triangle taken in order, the tangents to the circumscribing circle at the angles of the triangle are parallel respectively to

$$\alpha\beta\gamma, \quad \beta\gamma\alpha, \quad \text{and} \quad \gamma\alpha\beta.$$

Suppose two of these to be parallel, i. e. let

$$\alpha\beta\gamma = x\beta\gamma\alpha = x\alpha\gamma\beta \ (\S\ 90),$$

since the expression is a vector. Hence

$$\beta\gamma = x\gamma\beta,$$

which requires either

$$x = 1, \quad V\gamma\beta = 0 \quad \text{or} \quad \gamma \parallel \beta,$$

a case not contemplated in the problem;

or $$x = -1, \qquad S\beta\gamma = 0,$$

i. e. the triangle is right-angled. And geometry shews us at once that this is correct.

Again, if the triangle be isosceles, the tangent at the vertex is parallel to the base. Here we have

$$x\beta = \alpha\beta\gamma,$$

or $$x(\alpha + \gamma) = \alpha(\alpha + \gamma)\gamma;$$

whence $x = \gamma^2 = \alpha^2$, or $T\gamma = T\alpha$, as required.

As an elegant extension of this proposition the reader may prove that the vector of the continued product $\alpha\beta\gamma\delta$ of the vector-sides of any quadrilateral inscribed in a sphere is parallel to the radius drawn to the corner $(\alpha, \delta)$. [For, if $\epsilon$ be the vector from $\delta$, $\alpha$ to $\beta$, $\gamma$, $\alpha\beta\epsilon$ and $\epsilon\gamma\delta$ are (by what precedes) vectors *touching* the sphere at $\alpha$, $\delta$. And their product (whose vector part must be parallel to the radius at $\alpha$, $\delta$) is

$$\alpha\beta\epsilon \,.\, \epsilon\gamma\delta = \epsilon^2 \,.\, \alpha\beta\gamma\delta.]$$

107. To exemplify the variety of possible transformations even of simple expressions, we will take cases which are of frequent occurrence in applications to geometry.

Thus $$T(\rho+\alpha) = T(\rho-\alpha),$$
[which expresses that if

$$\overline{OA} = \alpha, \quad \overline{OA'} = -\alpha, \quad \text{and} \quad \overline{OP} = \rho,$$

we have $$AP = A'P,$$
and thus that $P$ is any point equidistant from two fixed points,] may be written $$(\rho+\alpha)^2 = (\rho-\alpha)^2,$$
or $$\rho^2 + 2S\alpha\rho + \alpha^2 = \rho^2 - 2S\alpha\rho + \alpha^2 \quad (\S\ 104),$$
whence $$S\alpha\rho = 0.$$
This may be changed to

$$\alpha\rho + \rho\alpha = 0,$$

or $$\alpha\rho + K\alpha\rho = 0,$$

$$SU\frac{\rho}{\alpha} = 0,$$

or finally, $$TVU\frac{\rho}{\alpha} = 1,$$
all of which express properties of a plane.

Again, $$T\rho = T\alpha$$
may be written $$T\frac{\rho}{\alpha} = 1,$$

$$\left(S\frac{\rho}{\alpha}\right)^2 - \left(V\frac{\rho}{\alpha}\right)^2 = 1,$$

$$(\rho+\alpha)^2 - 2S\alpha\,(\rho+\alpha) = 0,$$

$$\rho = (\rho+\alpha)^{-1}\alpha(\rho+\alpha),$$

$$S\,(\rho+\alpha)\,(\rho-\alpha) = 0,$$

or finally, $$T\,.\,(\rho+\alpha)\,(\rho-\alpha) = 2TV\alpha\rho.$$

All of these express properties of a sphere. They will be interpreted when we come to geometrical applications.

108. *To find the space relation among five points.*

A system of five points, so far as its internal relations are concerned, is fully given by the vectors from one to the other four. If three of these be called $\alpha$, $\beta$, $\gamma$, the fourth, $\delta$, is necessarily expressible as $x\alpha + y\beta + z\gamma$. Hence the relation required must be independent of $x$, $y$, $z$.

But
$$\left.\begin{aligned} S\alpha\delta &= x\alpha^2 + yS\alpha\beta + zS\alpha\gamma \\ S\beta\delta &= xS\beta\alpha + y\beta^2 + zS\beta\gamma \\ S\gamma\delta &= xS\gamma\alpha + yS\gamma\beta + z\gamma^2 \\ S\delta\delta = \delta^2 &= xS\delta\alpha + yS\delta\beta + zS\delta\gamma \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1).$$

The elimination of $x$, $y$, $z$ gives a determinant of the fourth order, which may be written

$$\begin{vmatrix} S\alpha\alpha & S\alpha\beta & S\alpha\gamma & S\alpha\delta \\ S\beta\alpha & S\beta\beta & S\beta\gamma & S\beta\delta \\ S\gamma\alpha & S\gamma\beta & S\gamma\gamma & S\gamma\delta \\ S\delta\alpha & S\delta\beta & S\delta\gamma & S\delta\delta \end{vmatrix} = 0.$$

Now each term may be put in either of two forms, thus

$$S\beta\gamma = \tfrac{1}{2}\{\beta^2 + \gamma^2 - (\beta-\gamma)^2\} = -T\beta T\gamma\cos\widehat{\beta\gamma}.$$

If the former be taken we have the expression connecting the distances, two and two, of five points in the form given by Muir (*Proc. R. S. E.* 1889); if we use the latter, the tensors divide out (some in rows, some in columns), and we have the relation among the cosines of the sides and diagonals of a spherical quadrilateral.

We may easily shew (as an exercise in quaternion manipulation merely) that this is the *only* condition, by shewing that from it we can get the condition when any other of the points is taken as origin. Thus, let the origin be at $\alpha$, the vectors are $-\alpha$, $\beta-\alpha$, $\gamma-\alpha$, $\delta-\alpha$. But, by changing the signs of the first row, and first column, of the determinant above, and then adding their values term by term to the other rows and columns, it becomes

$$\begin{vmatrix} S(\ -\alpha)(-\alpha) & S(\ -\alpha)(\beta-\alpha) & S(\ -\alpha)(\gamma-\alpha) & S(\ -\alpha)(\delta-\alpha) \\ S(\beta-\alpha)(-\alpha) & S(\beta-\alpha)(\beta-\alpha) & S(\beta-\alpha)(\gamma-\alpha) & S(\beta-\alpha)(\delta-\alpha) \\ S(\gamma-\alpha)(-\alpha) & S(\gamma-\alpha)(\beta-\alpha) & S(\gamma-\alpha)(\gamma-\alpha) & S(\gamma-\alpha)(\delta-\alpha) \\ S(\delta-\alpha)(-\alpha) & S(\delta-\alpha)(\beta-\alpha) & S(\delta-\alpha)(\gamma-\alpha) & S(\delta-\alpha)(\delta-\alpha) \end{vmatrix}$$

which, when equated to zero, gives the same relation as before. [See Ex. 10 at the end of this Chapter.]

An additional point, with $\epsilon = x'\alpha + y'\beta + z'\gamma$, gives *six* additional equations like (1); i. e.

$$\begin{aligned} S\alpha\epsilon &= x'\alpha^2 \quad + y'S\alpha\beta + z'S\alpha\gamma, \\ S\beta\epsilon &= x'S\beta\alpha + y'\beta^2 \quad + z'S\beta\gamma, \\ S\gamma\epsilon &= x'S\gamma\alpha + y'S\gamma\beta + z'\gamma^2, \\ S\delta\epsilon &= x'S\delta\alpha + y'S\delta\beta + z'S\delta\gamma \\ &= xS\epsilon\alpha \ + yS\epsilon\beta \ + zS\epsilon\gamma, \\ \epsilon^2 &= x'S\alpha\epsilon \ + y'S\beta\epsilon + z'S\gamma\epsilon, \end{aligned}$$

from which corresponding conclusions may be drawn.

Another mode of solving the problem at the head of this section is to write the *identity*

$$\Sigma m\,(\alpha - \theta)^2 = \Sigma m\alpha^2 - 2S\,.\,\theta\Sigma m\alpha + \theta^2\Sigma m,$$

where the $m$s are undetermined scalars, and the $\alpha$s are given vectors, while $\theta$ is any vector whatever.

Now, *provided that the number of given vectors exceeds four*, we do not completely determine the $m$s by imposing the conditions

$$\Sigma m = 0, \quad \Sigma m\alpha = 0.$$

Thus we may write the above identity, for each of five vectors successively, as

$$\begin{aligned} \Sigma m\,(\alpha - \alpha_1)^2 &= \Sigma m\alpha^2, \\ \Sigma m\,(\alpha - \alpha_2)^2 &= \Sigma m\alpha^2, \\ \ldots\ldots\ldots\ldots\ldots &= \ldots\ldots \\ \Sigma m\,(\alpha - \alpha_5)^2 &= \Sigma m\alpha^2. \end{aligned}$$

Take, with these, $$\Sigma m = 0,$$

and we have six linear equations from which to eliminate the $m$s. The resulting determinant is

$$\left|\begin{array}{cccccc} \overline{\alpha_1-\alpha_1}^2 & \overline{\alpha_1-\alpha_2}^2 & \overline{\alpha_1-\alpha_3}^2 & \ldots & \overline{\alpha_1-\alpha_5}^2 & 1 \\ \overline{\alpha_2-\alpha_1}^2 & \overline{\alpha_2-\alpha_2}^2 & \overline{\alpha_2-\alpha_3}^2 & \ldots & \overline{\alpha_2-\alpha_5}^2 & 1 \\ \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & & \cdot & \\ \overline{\alpha_5-\alpha_1}^2 & \overline{\alpha_5-\alpha_2}^2 & \cdot & \cdot & \overline{\alpha_5-\alpha_5}^2 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 0 \end{array}\right| \Sigma m\alpha^2 = 0.$$

This is equivalent to the form in which Cayley gave the relation among the mutual distances of five points. (*Camb. Math. Journ.* 1841.)

109. We have seen in § 95 that a quaternion may be divided into its scalar and vector parts as follows:—

$$\frac{\beta}{\alpha} = S\frac{\beta}{\alpha} + V\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}(\cos\theta + \epsilon\sin\theta);$$

where $\theta$ is the angle between the directions of $\alpha$ and $\beta$, and $\epsilon = UV\frac{\beta}{\alpha}$ is the unit-vector perpendicular to the plane of $\alpha$ and $\beta$ so situated that positive (i. e. left-handed) rotation about it turns $\alpha$ towards $\beta$.

Similarly we have (§ 96)

$$\begin{aligned} \alpha\beta &= S\alpha\beta + V\alpha\beta \\ &= T\alpha T\beta(-\cos\theta + \epsilon\sin\theta), \end{aligned}$$

$\theta$ and $\epsilon$ having the same signification as before.

110. Hence, considering the versor parts alone, we have

$$U\frac{\beta}{\alpha} = \cos\theta + \epsilon\sin\theta.$$

Similarly $$U\frac{\gamma}{\beta} = \cos\phi + \epsilon\sin\phi;$$

$\phi$ being the positive angle between the directions of $\gamma$ and $\beta$, and $\epsilon$ the same vector as before, if $\alpha$, $\beta$, $\gamma$ be coplanar.

Also we have

$$U\frac{\gamma}{\alpha} = \cos(\theta + \phi) + \epsilon\sin(\theta + \phi).$$

But we have always

$$\frac{\gamma}{\beta}.\frac{\beta}{\alpha} = \frac{\gamma}{\alpha},$$

and therefore $$U\frac{\gamma}{\beta}.U\frac{\beta}{\alpha} = U\frac{\gamma}{\alpha};$$

or $$\begin{aligned} \cos(\phi + \theta) + \epsilon\sin(\phi + \theta) &= (\cos\phi + \epsilon\sin\phi)(\cos\theta + \epsilon\sin\theta) \\ &= \cos\phi\cos\theta - \sin\phi\sin\theta + \epsilon(\sin\phi\cos\theta + \cos\phi\sin\theta), \end{aligned}$$

from which we have at once the fundamental formulae for the cosine and sine of the sum of two arcs, by equating separately the scalar and vector parts of these quaternions.

And we see, as an immediate consequence of the expressions above, that

$$\cos m\theta + \epsilon\sin m\theta = (\cos\theta + \epsilon\sin\theta)^m.$$

if $m$ be a positive whole number. For the left-hand side is a versor which turns through the angle $m\theta$ at once, while the right-hand side is a versor which effects the same object by $m$ successive turnings each through an angle $\theta$. See §§ 8, 9.

111. To extend this proposition to fractional indices we have only to write $\frac{\theta}{n}$ for $\theta$, when we obtain the results as in ordinary trigonometry.

From De Moivre's Theorem, thus proved, we may of course deduce the rest of Analytical Trigonometry. And as we have already deduced, as interpretations of self-evident quaternion transformations (§§ 97, 104), the fundamental formulae for the solution of plane triangles, we will now pass to the consideration of spherical trigonometry, a subject specially adapted for treatment by quaternions; but to which we cannot afford more than a very few sections. (More on this subject will be found in Chap. XI. in connexion with the Kinematics of rotation.) The reader is referred to Hamilton's works for the treatment of this subject by quaternion exponentials.

112. Let $\alpha$, $\beta$, $\gamma$ be unit-vectors drawn from the centre to the corners $A$, $B$, $C$ of a triangle on the unit-sphere. Then it is evident that, with the usual notation, we have (§ 96),

$$S\alpha\beta = -\cos c, \qquad S\beta\gamma = -\cos a, \qquad S\gamma\alpha = -\cos b,$$

$$TV\alpha\beta = \sin c, \quad TV\beta\gamma = \sin a, \quad TV\gamma\alpha = \sin b.$$

Also $UV\alpha\beta$, $UV\beta\gamma$, $UV\gamma\alpha$ are evidently the vectors of the corners of the polar triangle.

Hence
$$S\,.\,UV\alpha\beta UV\beta\gamma = \cos B, \text{ \&c.},$$
$$TV.\,UV\alpha\beta UV\beta\gamma = \sin B, \text{ \&c.}$$

Now (§ 90 (1)) we have

$$\begin{aligned} SV\alpha\beta V\beta\gamma &= S\,.\,\alpha V(\beta V\beta\gamma) \\ &= -S\alpha\beta S\beta\gamma + \beta^2 S\alpha\gamma. \end{aligned}$$

Remembering that we have

$$SV\alpha\beta V\beta\gamma = TV\alpha\beta TV\beta\gamma S\,.\,UV\alpha\beta UV\beta\gamma,$$

we see that the formula just written is equivalent to

$$\sin a \sin c \cos B = -\cos a \cos c + \cos b,$$

or $$\cos b = \cos a \cos c + \sin a \sin c \cos B.$$

113. Again, $V.V\alpha\beta V\beta\gamma = -\beta S\alpha\beta\gamma$,
which gives

$$TV.V\alpha\beta V\beta\gamma = TS.\alpha\beta\gamma = TS.\alpha V\beta\gamma = TS.\beta V\gamma\alpha = TS.\gamma V\alpha\beta,$$

or $\sin a \sin c \sin B = \sin a \sin p_a = \sin b \sin p_b = \sin c \sin p_c$;
where $p_a$ is the arc drawn from $A$ perpendicular to $BC$, &c.

Hence

$$\sin p_a = \sin c \sin B,$$

$$\sin p_b = \frac{\sin a \sin c}{\sin b} \sin B,$$

$$\sin p_c = \sin a \sin B.$$

114. Combining the results of the last two sections, we have

$$V\alpha\beta.V\beta\gamma = \sin a \sin c \cos B - \beta \sin a \sin c \sin B$$
$$= \sin a \sin c (\cos B - \beta \sin B).$$

Hence and
$$\left.\begin{aligned} U.V\alpha\beta V\beta\gamma &= (\cos B - \beta \sin B) \\ U.V\gamma\beta V\beta\alpha &= (\cos B + \beta \sin B) \end{aligned}\right\}.$$

These are therefore versors which turn all vectors perpendicular to $OB$ negatively or positively about $OB$ through the angle $B$.

[It will be shewn later (§ 119) that, in the combination

$$(\cos B + \beta \sin B)(\quad)(\cos B - \beta \sin B),$$

the system operated on is made to rotate, as if rigid, round the vector axis $\beta$ through an angle $2B$.]

As another instance, we have

$$\tan B = \frac{\sin B}{\cos B}$$

$$= \frac{TV.V\alpha\beta V\beta\gamma}{S.V\alpha\beta V\beta\gamma}$$

$$= -\beta^{-1}\frac{V.V\alpha\beta V\beta\gamma}{S.V\alpha\beta V\beta\gamma}$$

$$= -\frac{S.\alpha\beta\gamma}{S\alpha\gamma + S\alpha\beta S\beta\gamma} = \&c. \quad\ldots\ldots\ldots\ldots\ldots(1).$$

The interpretation of each of these forms gives a different theorem in spherical trigonometry.

115. Again, let us square the equal quantities

$$V.\alpha\beta\gamma \quad \text{and} \quad \alpha S\beta\gamma - \beta S\alpha\gamma + \gamma S\alpha\beta,$$

supposing $\alpha$, $\beta$, $\gamma$ to be any unit-vectors whatever. We have

$$-(V.\alpha\beta\gamma)^2 = S^2\beta\gamma + S^2\gamma\alpha + S^2\alpha\beta + 2S\beta\gamma S\gamma\alpha S\alpha\beta.$$

But the left-hand member may be written as

$$T^2.\alpha\beta\gamma - S^2.\alpha\beta\gamma,$$

whence

$$1 - S^2.\alpha\beta\gamma = S^2\beta\gamma + S^2\gamma\alpha + S^2\alpha\beta + 2S\beta\gamma S\gamma\alpha S\alpha\beta,$$

or $$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$$

$$= \sin^2 a \sin^2 p_a = \&c.$$

$$= \sin^2 a \sin^2 b \sin^2 C = \&c.,$$

all of which are well-known formulae.

116. Again, for any quaternion,

$$q = Sq + Vq,$$

so that, if $n$ be a positive integer,

$$q^n = (Sq)^n + n(Sq)^{n-1}Vq + \frac{n.\overline{n-1}}{1.2}(Sq)^{n-2}(Vq)^2 + \ldots$$

From this at once

$$S.q^n = (Sq)^n - \frac{n.\overline{n-1}}{1.2}(Sq)^{n-2}T^2Vq$$

$$+ \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}(Sq)^{n-4}T^4(Vq) - \&c.,$$

$$V.q^n = Vq\left[n(Sq)^{n-1} - \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}(Sq)^{n-3}T^2Vq + \&c.\right]$$

If $q$ be a versor we have

$$q = \cos u + \theta \sin u,$$

so that

$$S.q^n = (\cos u)^n - \frac{n.\overline{n-1}}{1.2}(\cos u)^{n-2}(\sin u)^2 + \ldots$$

$$= \cos nu;$$

$$V.q^n = \theta \sin u\left[n(\cos u)^{n-1} - \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}(\cos u)^{n-3}(\sin u)^2 + \ldots\right]$$

$$= \theta \sin nu;$$

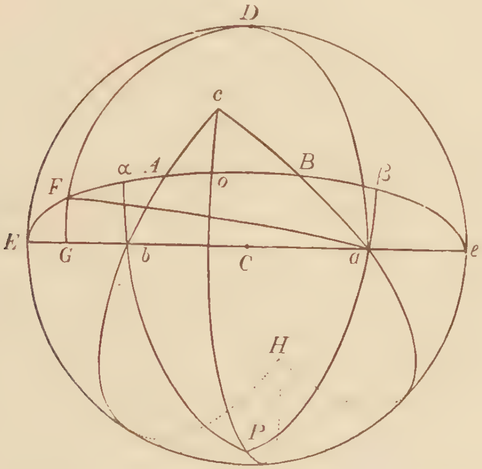
as we might at once have concluded from § 110.

Such results may be multiplied indefinitely by any one who has mastered the elements of quaternions.

117. A curious proposition, due to Hamilton, gives us a quaternion expression for the *spherical excess* in any triangle. The following proof, which is very nearly the same as one of his, though by no means the simplest that can be given, is chosen here because it incidentally gives a good deal of other information. We leave the quaternion proof as an exercise.

Let the unit-vectors drawn from the centre of the sphere to $A$, $B$, $C$, respectively, be $\alpha$, $\beta$, $\gamma$. It is required to express, as an arc and as an angle on the sphere, the quaternion

$$\beta\alpha^{-1}\gamma.$$



The figure represents an orthographic projection made on a plane perpendicular to $\gamma$. Hence $C$ is the centre of the circle $DEe$. Let the great circle through $A$, $B$ meet $DEe$ in $E$, $e$, and let $DE$ be a quadrant. Thus $\widehat{DE}$ represents $\gamma$ (§ 72). Also make $\widehat{EF} = \widehat{AB} = \beta\alpha^{-1}$. Then, evidently,

$$\widehat{DF} = \beta\alpha^{-1}\gamma,$$

which gives the arcual representation required.

Let $DF$ cut $Ee$ in $G$. Make $Ca = EG$, and join $D$, $a$, and $a$, $F$. Obviously, as $D$ is the pole of $Ee$, $Da$ is a quadrant; and since $EG = Ca$, $Ga = EC$, a quadrant also. Hence $a$ is the pole of $DG$, and therefore the quaternion may be represented by the angle $DaF$.

Make $Cb = Ca$, and draw the arcs $Pa\beta$, $Pb\alpha$ from $P$, the pole of

$AB$. Comparing the triangles $Eb\alpha$ and $ea\beta$, we see that $E\alpha = e\beta$. But, since $P$ is the pole of $AB$, $F\beta a$ is a right angle: and therefore as $Fa$ is a quadrant, so is $F\beta$. Thus $AB$ is the complement of $E\alpha$ or $\beta e$, and therefore

$$\alpha\beta = 2AB.$$

Join $bA$ and produce it to $c$ so that $Ac = bA$; join $c$, $P$, cutting $AB$ in $o$. Also join $c$, $B$, and $B$, $a$.

Since $P$ is the pole of $AB$, the angles at $o$ are right angles; and therefore, by the equal triangles $b\alpha A$, $coA$, we have

$$\alpha A = Ao.$$

But $$\alpha\beta = 2AB,$$

whence $$oB = B\beta,$$

and therefore the triangles $coB$ and $Ba\beta$ are equal, and $c$, $B$, $a$ lie on the same great circle.

Produce $cA$ and $cB$ to meet in $H$ (on the opposite side of the sphere). $H$ and $c$ are diametrically opposite, and therefore $cP$, produced, passes through $H$.

Now $Pa = Pb = PH$, for they differ from quadrants by the equal arcs $a\beta$, $b\alpha$, $oc$. Hence these arcs divide the triangle $Hab$ into three isosceles triangles.

But $$\angle PHb + \angle PHa = \angle aHb = \angle bca.$$

Also $$\angle Pab = \pi - \angle cab - \angle PaH,$$

$$\angle Pba = \angle Pab = \pi - \angle cba - \angle PbH.$$

Adding, $$2\angle Pab = 2\pi - \angle cab - \angle cba - \angle bca$$

$$= \pi - (\text{spherical excess of } abc).$$

But, as $\angle Fa\beta$ and $\angle Dae$ are right angles, we have

$$\text{angle of } \beta\alpha^{-1}\gamma = \angle FaD = \angle \beta ae = \angle Pab$$

$$= \frac{\pi}{2} - \tfrac{1}{2}(\text{spherical excess of } abc).$$

[Numerous singular geometrical theorems, easily proved *ab initio* by quaternions, follow from this: e.g. The arc $AB$, which bisects two sides of a spherical triangle $abc$, intersects the base at the distance of a quadrant from its middle point. All spherical triangles, with a common side, and having their other sides bisected by the same great circle (i.e. having their vertices in a small circle parallel to this great circle) have equal areas, &c. &c.]

118. Let $\overline{Oa} = \alpha'$, $\overline{Ob} = \beta'$, $\overline{Oc} = \gamma'$, and we have

$$\left(\frac{\alpha'}{\beta'}\right)^{\frac{1}{2}}\left(\frac{\beta'}{\gamma'}\right)^{\frac{1}{2}}\left(\frac{\gamma'}{\alpha'}\right)^{\frac{1}{2}} = \overset{\frown}{Ca}\,.\,\overset{\frown}{cA}\,.\,\overset{\frown}{Bc}$$

$$= \overset{\frown}{Ca}\,.\,\overset{\frown}{BA}$$

$$= \overset{\frown}{EG}\,.\,\overset{\frown}{FE} = \overset{\frown}{FG}.$$

But $FG$ is the complement of $DF$. Hence the *angle of the quaternion*

$$\left(\frac{\alpha'}{\beta'}\right)^{\frac{1}{2}}\left(\frac{\beta'}{\gamma'}\right)^{\frac{1}{2}}\left(\frac{\gamma'}{\alpha'}\right)^{\frac{1}{2}}$$

*is half the spherical excess of the triangle whose angular points are at the extremities of the unit-vectors* $\alpha'$, $\beta'$, $\gamma'$.

[In seeking a purely quaternion proof of the preceding propositions, the student may commence by shewing that for any three unit-vectors we have

$$\frac{\beta}{\alpha}\frac{\gamma}{\beta}\frac{\alpha}{\gamma} = -(\beta\alpha^{-1}\gamma)^2.$$

The angle of the first of these quaternions can be easily assigned; and the equation shews how to find that of $\beta\alpha^{-1}\gamma$.

Another easy method is to commence afresh by forming from the vectors of the corners of a spherical triangle three new vectors thus:—

$$\alpha' = \left(\frac{\beta+\gamma}{\alpha}\right)^2 . \alpha, \text{ \&c.}$$

Then the angle between the planes of $\alpha$, $\beta'$ and $\gamma'$, $\alpha$; or of $\beta$, $\gamma'$ and $\alpha'$, $\beta$; or of $\gamma$, $\alpha'$ and $\beta'$, $\gamma$; is obviously the spherical excess.

But a still simpler method of proof is easily derived from the composition of rotations.]

119. It may be well to introduce here, though it belongs rather to Kinematics than to Geometry, the interpretation of the operator

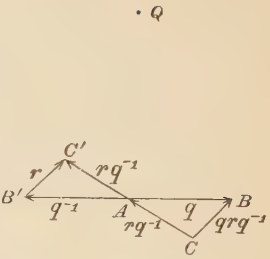
$$q(\quad)q^{-1}.$$

By a rotation, about the axis of $q$, through double the angle of $q$, the quaternion $r$ becomes the quaternion $qrq^{-1}$. Its tensor and angle remain unchanged, its plane or axis alone varies.

A glance at the figure is sufficient for the proof, if we note that of course $T\,.\,qrq^{-1}=Tr$, and therefore that we need consider the *versor* parts only. Let $Q$ be the pole of $q$,

$$\widehat{AB}=q,\quad \widehat{AB'}=q^{-1},\quad \widehat{B'C'}=r.$$

Join $C'A$, and make $\widehat{AC}=\widehat{C'A}$. Join $CB$.



Then $\widehat{CB}$ is $qrq^{-1}$, its arc $CB$ is evidently equal in length to that of $r$, $B'C'$; and its plane (making the same angle with $B'B$ that that of $B'C'$ does) has evidently been made to revolve about $Q$, the pole of $q$, through double the angle of $q$.

It is obvious, from the nature of the above proof, that this operation is distributive; i.e. that

$$q\,(r+s)\,q^{-1}=qrq^{-1}+qsq^{-1}.$$

If $r$ be a vector, $=\rho$, then $q\rho q^{-1}$ (which is also a vector) is the result of a rotation through double the angle of $q$ about the axis of $q$. Hence, as Hamilton has expressed it, if $B$ represent a rigid system, or assemblage of vectors,

$$qBq^{-1}$$

is its new position after rotating through double the angle of $q$ about the axis of $q$.

120. To compound such rotations, we have

$$r\,.\,qBq^{-1}\,.\,r^{-1}=rq\,.\,B\,.\,(rq)^{-1}.$$

To cause rotation through an angle $t$-fold the double of the angle of $q$ we write $$q^tBq^{-t}.$$

To reverse the direction of this rotation write $q^{-t}Bq^t$.

To *translate* the body $B$ without rotation, each point of it moving through the vector $\alpha$, we write $\alpha+B$.

To produce rotation of the translated body about the same axis, and through the same angle, as before,

$$q\,(\alpha+B)\,q^{-1}.$$

Had we rotated first, and then translated, we should have had

$$\alpha+qBq^{-1}.$$

From the point of view of those who do not believe in the Moon's rotation, the former of these expressions ought to be

$$q\alpha q^{-1}+B$$

instead of

$$q\alpha q^{-1}+qBq^{-1}.$$

But to such men quaternions are unintelligible.

121. The operator above explained finds, of course, some of its most direct applications in the ordinary questions of Astronomy, connected with the apparent diurnal rotation of the stars. If $\lambda$ be a unit-vector parallel to the polar axis, and $h$ the hour angle from the meridian, the operator is

$$\left(\cos\frac{h}{2}-\lambda\sin\frac{h}{2}\right)\left(\quad\right)\left(\cos\frac{h}{2}+\lambda\sin\frac{h}{2}\right),$$

or

$$L^{-1}(\quad)L,$$

the inverse going first, because the *apparent* rotation is negative (clockwise).

If the upward line be $i$, and the southward $j$, we have

$$\lambda=i\sin l-j\cos l,$$

where $l$ is the latitude of the observer. The meridian equatorial unit vector is

$$\mu=i\cos l+j\sin l;$$

and $\lambda$, $\mu$, $k$ of course form a rectangular unit system.

The meridian unit-vector of a heavenly body is

$$\begin{aligned}\delta&=i\cos(l-d)+j\sin(l-d),\\&=\lambda\sin d+\mu\cos d,\end{aligned}$$

where $d$ is its declination.

Hence when its hour-angle is $h$, its vector is

$$\delta'=L^{-1}\delta L.$$

The vertical plane containing it intersects the horizon in

$$iVi\delta'=jSj\delta'+kSk\delta',$$

so that

$$\tan\text{(azimuth)}=\frac{Sk\delta'}{Sj\delta'}\quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

[This may also be obtained directly from the last formula (1) of § 114.]

To find its Amplitude, i.e. its azimuth at rising or setting, the hour-angle must be obtained from the condition

$$Si\delta' = 0 \quad\dots\dots\dots\dots(2).$$

These relations, with others immediately deducible from them, enable us (at once and for ever) to dispense with the hideous formulae of Spherical Trigonometry.

122. To shew how readily they can be applied, let us translate the expressions above into the ordinary notation. This is effected at once by means of the expressions for $\lambda$, $\mu$, $L$, and $\delta$ above, which give by inspection

$$\delta' = \lambda \sin d + (\mu \cos h - k \sin h) \cos d,$$

and we have from (1) and (2) of last section respectively

$$\tan(\text{azimuth}) = \frac{\sin h \cos d}{\cos l \sin d - \sin l \cos d \cos h} \quad\dots\dots\dots(1),$$

$$\cos h + \tan l \tan d = 0 \quad\dots\dots\dots(2).$$

In Capt. Weir's ingenious *Azimuth Diagram*, these equations are represented graphically by the rectangular cöordinates of a system of confocal conics:—viz.

$$\left.\begin{aligned} x &= \sin h \sec l \\ y &= \cos h \tan l \end{aligned}\right\} \quad\dots\dots\dots(3).$$

The ellipses of this system depend upon $l$ alone, the hyperbolas upon $h$. Since (1) can, by means of (3), be written as

$$\tan(\text{azimuth}) = \frac{x}{\tan d - y},$$

we see that the azimuth can be constructed at once by joining with the point $0, -\tan d$, the intersection of the proper ellipse and hyperbola.

Equation (2) puts these expressions for the cöordinates in the form

$$\left.\begin{aligned} x &= \sec l \sqrt{1 - \tan^2 l \tan^2 d} \\ y &= -\tan^2 l \tan d \end{aligned}\right\}.$$

The elimination of $d$ gives the ellipse as before, but that of $l$ gives, instead of the hyperbolas, the circles

$$x^2 + y^2 - y(\tan d - \cot d) = 1.$$

The radius is

$$\tfrac{1}{2}(\tan d + \cot d);$$

and the cöordinates of the centre are

$$0,\ \tfrac{1}{2}(\tan d - \cot d).$$

123. A *scalar* equation in $\rho$, the vector of an undetermined point, is generally the equation of a *surface*; since we may use in it the expression

$$\rho = x\alpha,$$

where $x$ is an unknown scalar, and $\alpha$ any assumed unit-vector. The result is an equation to determine $x$. Thus one or more points are found on the vector $x\alpha$, whose coördinates satisfy the equation; and the locus is a surface whose degree is determined by that of the equation which gives the values of $x$.

But a *vector* equation in $\rho$, as we have seen, generally leads to three scalar equations, from which the three rectangular or other components of the sought vector are to be derived. Such a vector equation, then, usually belongs to a definite number of *points* in space. But in certain cases these may form a *line*, and even a *surface*, the vector equation losing as it were one or two of the three scalar equations to which it is usually equivalent.

Thus while the equation $\quad \alpha\rho = \beta$

gives at once $\quad \rho = \alpha^{-1}\beta,$

which is the vector of a definite point (since by making $\rho$ a *vector* we have evidently assumed

$$S\alpha\beta = 0);$$

the closely allied equation $\quad V\alpha\rho = \beta$

is easily seen to involve $\quad S\alpha\beta = 0,$

and to be satisfied by $\quad \rho = \alpha^{-1}\beta + x\alpha,$

whatever be $x$. Hence the vector of any point whatever in the line drawn parallel to $\alpha$ from the extremity of $\alpha^{-1}\beta$ satisfies the given equation. [The difference between the results depends upon the fact that $S\alpha\rho$ is indeterminate in the second form, but definite $(=0)$ in the first.]

124. Again, $\quad V\alpha\rho \,.\, V\rho\beta = (V\alpha\beta)^2$

is equivalent to but two scalar equations. For it shews that $V\alpha\rho$ and $V\beta\rho$ are parallel, i.e. $\rho$ lies in the same plane as $\alpha$ and $\beta$, and can therefore be written (§ 24)

$$\rho = x\alpha + y\beta,$$

where $x$ and $y$ are scalars as yet undetermined.

We have now

$$V\alpha\rho = yV\alpha\beta,$$
$$V\rho\beta = xV\alpha\beta,$$

which, by the given equation, lead to

$$xy=1, \quad \text{or} \quad y=\frac{1}{x},$$

or finally
$$\rho=x\alpha+\frac{1}{x}\beta;$$

which (§ 40) is the equation of a hyperbola whose asymptotes are in the directions of $\alpha$ and $\beta$.

125. Again, the equation

$$V.V\alpha\beta V\alpha\rho=0,$$

though apparently equivalent to three scalar equations, is really equivalent to one only. In fact we see by § 91 that it may be written

$$-\alpha S.\alpha\beta\rho=0,$$

whence, if $\alpha$ be not zero, we have

$$S.\alpha\beta\rho=0,$$

and thus (§ 101) the only condition is that $\rho$ is coplanar with $\alpha$, $\beta$. Hence the equation represents the plane in which $\alpha$ and $\beta$ lie.

126. Some very curious results are obtained when we extend these processes of interpretation to functions of a *quaternion*

$$q=w+\rho$$

instead of functions of a mere *vector* $\rho$.

A scalar equation containing such a quaternion, along with quaternion constants, gives, as in last section, the equation of a surface, if we assign a definite value to $w$. Hence for successive values of $w$, we have successive surfaces belonging to a system; and thus when $w$ is indeterminate the equation represents not a *surface*, as before, but a *volume*, in the sense that the vector of any point within that volume satisfies the equation.

Thus the equation
$$(Tq)^2=a^2,$$
or
$$w^2-\rho^2=a^2,$$
or
$$(T\rho)^2=a^2-w^2,$$

represents, for any assigned value of $w$, not greater than $a$, a sphere whose radius is $\sqrt{a^2-w^2}$. Hence the equation is satisfied by the vector of any point whatever in the *volume* of a sphere of radius $a$, whose centre is origin.

Again, by the same kind of investigation,

$$(T(q-\beta))^2=a^2,$$

where $q=w+\rho$, is easily seen to represent the volume of a sphere of radius $a$ described about the extremity of $\beta$ as centre.

Also $S(q^2)=-a^2$ is the equation of infinite space less the space contained in a sphere of radius $a$ about the origin.

Similar consequences as to the interpretation of vector equations in quaternions may be readily deduced by the reader.

127. The following transformation is enuntiated without proof by Hamilton (*Lectures*, p. 587, and *Elements*, p. 299).

$$r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1}=U(rq+KrKq).$$

To prove it, let $$r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1}=t,$$

then $$Tt=1,$$

and therefore $$Kt=t^{-1}.$$

But $$(r^2q^2)^{\frac{1}{2}}=rtq,$$

or $$r^2q^2=rtqrtq,$$

or $$rq=tqrt.$$

Hence $$KqKr=t^{-1}KrKqt^{-1},$$

or $$KrKq=tKqKrt.$$

Thus we have $$U(rq\pm KrKq)=tU(qr\pm KqKr)t,$$

or, if we put $$s=U(qr\pm KqKr),$$

$$Ks=\pm tst.$$

Hence $$sKs=(Ts)^2=1=\pm stst,$$

which, if we take the positive sign, requires

$$st=\pm 1,$$

or $$t=\pm s^{-1}=\pm UKs,$$

which is the required transformation.

[It is to be noticed that there are other results which might have been arrived at by using the negative sign above; some involving an arbitrary unit-vector, others involving the imaginary of ordinary algebra.]

128. As a final example, we take a transformation of Hamilton's, of great importance in the theory of surfaces of the second order.

Transform the expression

$$(S\alpha\rho)^2+(S\beta\rho)^2+(S\gamma\rho)^2$$

in which $\alpha, \beta, \gamma$ are any three mutually rectangular vectors, into the form

$$\left(\frac{T(\iota\rho+\rho\kappa)}{\kappa^2-\iota^2}\right)^2,$$

which involves only two vector-constants, $\iota$, $\kappa$.

[The student should remark here that $\iota$, $\kappa$, two undetermined vectors, involve six disposable constants:—and that $\alpha$, $\beta$, $\gamma$, being a *rectangular* system, involve also only six constants.]

$$\begin{aligned}\{T(\iota\rho+\rho\kappa)\}^2 &= (\iota\rho+\rho\kappa)(\rho\iota+\kappa\rho) \quad (\S\S\ 52,\ 55)\\ &= (\iota^2+\kappa^2)\rho^2+(\iota\rho\kappa\rho+\rho\kappa\rho\iota)\\ &= (\iota^2+\kappa^2)\rho^2+2S.\iota\rho\kappa\rho\\ &= (\iota-\kappa)^2\rho^2+4S\iota\rho S\kappa\rho.\end{aligned}$$

Hence $\quad (S\alpha\rho)^2+(S\beta\rho)^2+(S\gamma\rho)^2 = \dfrac{(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2}\rho^2+4\dfrac{S\iota\rho S\kappa\rho}{(\kappa^2-\iota^2)^2}.$

But $\quad \alpha^{-2}(S\alpha\rho)^2+\beta^{-2}(S\beta\rho)^2+\gamma^{-2}(S\gamma\rho)^2=\rho^2 \quad (\S\S\ 25,\ 73).$

Multiply by $\beta^2$ and subtract, we get

$$\left(1-\frac{\beta^2}{\alpha^2}\right)(S\alpha\rho)^2-\left(\frac{\beta^2}{\gamma^2}-1\right)(S\gamma\rho)^2=\left\{\frac{(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2}-\beta^2\right\}\rho^2+4\frac{S\iota\rho S\kappa\rho}{(\kappa^2-\iota^2)^2}.$$

The left side breaks up into two real factors if $\beta^2$ be intermediate in value to $\alpha^2$ and $\gamma^2$: and that the right side may do so the term in $\rho^2$ must vanish. This condition gives

$$\beta^2=\frac{(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2};\text{ and the identity becomes}$$

$$S\left\{\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}+\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right\}\rho S\left\{\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}-\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right\}\rho$$
$$=4\frac{S\iota\rho S\kappa\rho}{(\kappa^2-\iota^2)^2}.$$

Hence we must have

$$\frac{2\iota}{\kappa^2-\iota^2}=p\left\{\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}+\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right\},$$

$$\frac{2\kappa}{\kappa^2-\iota^2}=\frac{1}{p}\left\{\alpha\sqrt{\left(1-\frac{\beta^2}{\alpha^2}\right)}-\gamma\sqrt{\left(\frac{\beta^2}{\gamma^2}-1\right)}\right\},$$

where $p$ is an undetermined scalar.

To determine $p$, substitute in the expression for $\beta^2$, and we find

$$\begin{aligned}4\beta^2=\frac{4(\iota-\kappa)^2}{(\kappa^2-\iota^2)^2}&=\left(p-\frac{1}{p}\right)^2(\alpha^2-\beta^2)+\left(p+\frac{1}{p}\right)^2(\beta^2-\gamma^2)\\ &=\left(p^2+\frac{1}{p^2}\right)(\alpha^2-\gamma^2)-2(\alpha^2+\gamma^2)+4\beta^2.\end{aligned}$$

Thus the transformation succeeds if

$$p^2 + \frac{1}{p^2} = \frac{2(\alpha^2+\gamma^2)}{\alpha^2-\gamma^2},$$

which gives

$$p + \frac{1}{p} = \pm 2\sqrt{\frac{\alpha^2}{\alpha^2-\gamma^2}},$$

$$p - \frac{1}{p} = \pm 2\sqrt{\frac{\gamma^2}{\alpha^2-\gamma^2}}.$$

Hence

$$\frac{4(\kappa^2-\iota^2)}{(\kappa^2-\iota^2)^2} = \left(\frac{1}{p^2}-p^2\right)(\alpha^2-\gamma^2) = \pm 4\sqrt{\alpha^2\gamma^2},$$

or

$$(\kappa^2-\iota^2)^{-1} = \pm T\alpha T\gamma.$$

Again,

$$p = \frac{T\alpha+T\gamma}{\sqrt{\gamma^2-\alpha^2}}, \quad \frac{1}{p} = \frac{T\alpha-T\gamma}{\sqrt{\gamma^2-\alpha^2}},$$

and therefore

$$2\iota = \frac{T\alpha+T\gamma}{T\alpha\, T\gamma}\left(\sqrt{\frac{\beta^2-\alpha^2}{\gamma^2-\alpha^2}}\,U\alpha + \sqrt{\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2}}\,U\gamma\right),$$

$$2\kappa = \frac{T\alpha-T\gamma}{T\alpha\, T\gamma}\left(\sqrt{\frac{\beta^2-\alpha^2}{\gamma^2-\alpha^2}}\,U\alpha - \sqrt{\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2}}\,U\gamma\right),$$

Thus we have proved the possibility of the transformation, and determined the transforming vectors $\iota$, $\kappa$.

129. By differentiating the equation

$$(S\alpha\rho)^2 + (S\beta\rho)^2 + (S\gamma\rho)^2 = \left(\frac{T(\iota\rho+\rho\kappa)}{(\kappa^2-\iota^2)}\right)^2$$

we obtain, as will be seen in Chapter IV, the following,

$$S\alpha\rho S\alpha\rho' + S\beta\rho S\beta\rho' + S\gamma\rho S\gamma\rho' = \frac{S.(\iota\rho+\rho\kappa)(\kappa\rho'+\rho'\iota)}{(\kappa^2-\iota^2)^2},$$

where $\rho'$ also may be any vector whatever.

This is another very important formula of transformation; and it will be a good exercise for the student to prove its truth by processes analogous to those in last section. We may merely observe, what indeed is obvious, that by putting $\rho' = \rho$ it becomes the formula of last section. And we see that we may write, with the recent values of $\iota$ and $\kappa$ in terms of $\alpha$, $\beta$ $\gamma$, the identity

$$\alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho = \frac{(\iota^2+\kappa^2)\rho + 2V.\iota\rho\kappa}{(\kappa^2-\iota^2)^2}$$

$$= \frac{(\iota-\kappa)^2\rho + 2(\iota S\kappa\rho + \kappa S\iota\rho)}{(\kappa^2-\iota^2)^2}.$$

130. In various quaternion investigations, especially in such as involve *imaginary* intersections of curves and surfaces, the old imaginary of algebra of course appears. But it is to be particularly noticed that this expression is analogous to a scalar and not to a vector, and that like real scalars it is commutative in multiplication with all other factors. Thus it appears, by the same proof as in algebra, that any quaternion expression which contains this imaginary can always be broken up into the sum of two parts, one real, the other multiplied by the first power of $\sqrt{-1}$. Such an expression, viz.

$$q = q' + \sqrt{-1}q'',$$

where $q'$ and $q''$ are real quaternions, is called by Hamilton a BIQUATERNION. [The student should be warned that the term Biquaternion has since been employed by other writers in the sense sometimes of a "set" of 8 elements, analogous to the Quaternion 4; sometimes for an expression $q' + \theta q''$ where $\theta$ is not the algebraic imaginary. By them Hamilton's Biquaternion is called simply a quaternion with non-real constituents.] Some little care is requisite in the management of these expressions, but there is no new difficulty. The points to be observed are: first, that any biquaternion can be divided into a real and an imaginary part, the latter being the product of $\sqrt{-1}$ by a real quaternion; second, that this $\sqrt{-1}$ is commutative with all other quantities in multiplication; third, that if two biquaternions be equal, as

$$q' + \sqrt{-1}\,q'' = r' + \sqrt{-1}\,r'',$$

we have, as in algebra, $q' = r'$, $q'' = r''$;

so that an equation between biquaternions involves in general *eight* equations between scalars. Compare § 80.

131. We have obviously, since $\sqrt{-1}$ is a scalar,

$$S(q' + \sqrt{-1}\,q'') = Sq' + \sqrt{-1}\,Sq'',$$
$$V(q' + \sqrt{-1}\,q'') = Vq' + \sqrt{-1}\,Vq''.$$

Hence (§ 103)

$$\begin{aligned}&\{T(q' + \sqrt{-1}\,q'')\}^2\\ &= (Sq' + \sqrt{-1}\,Sq'' + Vq' + \sqrt{-1}\,Vq'')\,(Sq' + \sqrt{-1}\,Sq'' - Vq' - \sqrt{-1}\,Vq'')\\ &= (Sq' + \sqrt{-1}\,Sq'')^2 - (Vq' + \sqrt{-1}\,Vq'')^2,\\ &= (Tq')^2 - (Tq'')^2 + 2\sqrt{-1}S\,.\,q'Kq''.\end{aligned}$$

The only remark which need be made on such formulae is this, that *the tensor of a biquaternion may vanish while both of the component quaternions are finite.*

Thus, if $$Tq' = Tq'',$$
and $$S.q'Kq'' = 0,$$
the above formula gives
$$T(q' + \sqrt{-1}\,q'') = 0.$$
The condition $$S.q'Kq'' = 0$$
may be written
$$Kq'' = q'^{-1}\alpha, \quad \text{or} \quad q'' = -\alpha Kq'^{-1} = -\frac{\alpha q'}{(Tq')^2},$$
where $\alpha$ is any vector whatever.

Hence $$Tq' = Tq'' = TKq'' = \frac{T\alpha}{Tq'},$$
and therefore
$$Tq'\,(Uq' - \sqrt{-1}\,U\alpha\,.\,Uq') = (1 - \sqrt{-1}\,U\alpha)\,q'$$
is the general form of a biquaternion whose tensor is zero.

132. More generally we have, $q$, $r$, $q'$, $r'$ being any four real and non-evanescent quaternions,
$$(q + \sqrt{-1}q')(r + \sqrt{-1}r') = qr - q'r' + \sqrt{-1}\,(qr' + q'r).$$
That this product may vanish we must have
$$qr = q'r',$$
and $$qr' = -q'r.$$
Eliminating $r'$ we have $$qq'^{-1}qr = -q'r,$$
which gives $$(q'^{-1}q)^2 = -1,$$
i.e. $$q = q'\alpha$$
where $\alpha$ is some unit-vector.

And the two equations now agree in giving
$$-r = \alpha r',$$
so that we have the biquaternion factors in the form
$$q'(\alpha + \sqrt{-1}) \quad \text{and} \quad -(\alpha - \sqrt{-1})\,r';$$
and their product is
$$-q'(\alpha + \sqrt{-1})(\alpha - \sqrt{-1})\,r',$$
which, of course, vanishes.

[A somewhat simpler investigation of the same proposition may be obtained by writing the biquaternions as

$$q'(q'^{-1}q+\sqrt{-1}) \quad \text{and} \quad (rr'^{-1}+\sqrt{-1})\,r',$$

or

$$q'(q''+\sqrt{-1}) \quad \text{and} \quad (r''+\sqrt{-1})\,r',$$

and shewing that

$$q''=-r''=\alpha, \text{ where } T\alpha=1.]$$

From this it appears that if the product of two *bivectors*

$$\rho+\sigma\sqrt{-1} \quad \text{and} \quad \rho'+\sigma'\sqrt{-1}$$

is zero, we must have

$$\sigma^{-1}\rho=-\rho'\sigma'^{-1}=U\alpha,$$

where $\alpha$ may be any vector whatever. But this result is still more easily obtained by means of a direct process.

133. It may be well to observe here (as we intend to avail ourselves of them in the succeeding Chapters) that certain abbreviated forms of expression may be used when they are not liable to confuse, or lead to error. Thus we may write

$$T^2q \text{ for } (Tq)^2,$$

just as we write

$$\cos^2\theta \text{ for } (\cos\theta)^2,$$

although the true meanings of these expressions are

$$T(Tq) \text{ and } \cos(\cos\theta).$$

The former is justifiable, as $T(Tq)=Tq$, and therefore $T^2q$ is not required to signify the second tensor (or tensor of the tensor) of $q$. But the trigonometrical usage is defensible only on the score of convenience, and is habitually violated by the employment of $\cos^{-1}x$ in its natural and proper sense.

Similarly we may write

$$S^2q \text{ for } (Sq)^2, \text{ \&c.},$$

but it may be advisable not to use

$$Sq^2$$

as the equivalent of either of those just written; inasmuch as it might be confounded with the (generally) different quantity

$$S.q^2 \text{ or } S(q^2),$$

although this is rarely written without the point or the brackets.

The question of the use of points or brackets is one on which no very definite rules can be laid down. A beginner ought to use them freely, and he will soon learn by trial which of them are absolutely necessary to prevent ambiguity.

In the present work this course has been adopted:—the earlier examples in each part of the subject being treated with a free use of points and brackets, while in the later examples superfluous marks of the kind are gradually got rid of.

It may be well to indicate some general principles which regulate the *omission* of these marks. Thus in $S.\alpha\beta$ or $V.\alpha\beta$ the point is obviously unnecessary:—because $S\alpha = 0$, and $V\alpha = \alpha$, so that the $S$ would annihilate the term if it applied to $\alpha$ alone, while in the same case the $V$ would be superfluous. But in $S.qr$ and $V.qr$, the point (or an equivalent) is indispensable, for $Sq.r$, and $Vq.r$ are usually quite different from the first written quantities. In the case of $K$, and of $d$ (used for scalar differentiation), the omission of the point indicates that the operator acts *only* on the nearest factor:—thus

$$Kqr = (Kq)\,r = Kq.r, \quad dqr = (dq)\,r = dq.r;$$

while, if its action extend farther, we write

$$K.qr = K\,(qr), \quad d.qr = d\,(qr), \text{ \&c.}$$

In more complex cases we must be ruled by the general principle of dropping nothing which is essential. Thus, for instance

$$V\,(pK\,(dq)\,V\,(Vq.r))$$

may be written without ambiguity as

$$V.pKdqVVqr,$$

but nothing more can be dropped without altering its value.

Another peculiarity of notation, which will occasionally be required, shows *which portions* of a complex product are affected by an operator. Thus we write

$$\nabla S\sigma\tau$$

if $\nabla$ operates on $\sigma$ and also on $\tau$, but

$$\nabla_1 S\sigma\tau_1$$

if it operates on $\tau$ alone. See, in this connection, the last Example at the end of Chap. IV. below.

134. The beginner may expect to be at first a little puzzled with this aspect of the notation; but, as he learns more of the subject, he will soon see clearly the distinction between such an expression as

$$S.V\alpha\beta V\beta\gamma,$$

where we may omit at pleasure either the point or the first $V$ without altering the value, and the very different one

$$S\alpha\beta \, . \, V\beta\gamma,$$

which admits of no such changes, without alteration of its value.

All these simplifications of notation are, in fact, merely examples of the transformations of quaternion expressions to which part of this Chapter has been devoted. Thus, to take a very simple example, we easily see that

$$\begin{aligned} S \, . \, V\alpha\beta \, V\beta\gamma &= SV\alpha\beta \, V\beta\gamma = S \, . \, \alpha\beta \, V\beta\gamma = S\alpha V \, . \, \beta \, V\beta\gamma = -S\alpha V \, . \, (V\beta\gamma)\beta \\ &= S\alpha V \, . \, (V\gamma\beta)\beta = S \, . \, \alpha V(\gamma\beta)\beta = S \, . \, V(\gamma\beta)\beta\alpha = SV\gamma\beta \, V\beta\alpha \\ &= S \, . \, \gamma\beta \, V\beta\alpha = S \, . \, K(\beta\gamma) \, V\beta\alpha = S \, . \, \beta\gamma K \, V\beta\alpha = -S \, . \, \beta\gamma \, V\beta\alpha \\ &= S \, . \, V\gamma\beta \, V\beta\alpha, \text{ \&c., \&c.} \end{aligned}$$

The above group does not nearly exhaust the list of even the simpler ways of expressing the given quantity. We recommend it to the careful study of the reader. He will find it advisable, at first, to use stops and brackets pretty freely; but will gradually learn to dispense with those which are not absolutely necessary to prevent ambiguity.

There is, however, one additional point of notation to which the reader's attention should be most carefully directed. A very simple instance will suffice. Take the expressions

$$\frac{\beta}{\gamma} \, . \, \frac{\gamma}{\alpha} \quad \text{and} \quad \frac{\beta\gamma}{\gamma\alpha}.$$

The first of these is

$$\beta\gamma^{-1} \, . \, \gamma\alpha^{-1} = \beta\alpha^{-1},$$

and presents no difficulty. But the second, though at first sight it closely resembles the first, is in general totally different in value, being in fact equal to

$$\beta\gamma\alpha^{-1}\gamma^{-1}.$$

For the denominator must be treated as *one quaternion.* If, then, we write

$$\frac{\beta\gamma}{\gamma\alpha} = q,$$

we have

$$\beta\gamma = q\gamma\alpha,$$

so that, as stated above,

$$q = \beta\gamma\alpha^{-1}\gamma^{-1}.$$

We see therefore that

$$\frac{\beta}{\gamma} \, . \, \frac{\gamma}{\alpha} = \frac{\beta}{\alpha} = \frac{\beta\gamma}{\alpha\gamma}; \text{ but } \textit{not} = \frac{\beta\gamma}{\gamma\alpha}.$$

## EXAMPLES TO CHAPTER III.

1. Investigate, by quaternions, the requisite formulae for changing from any one set of cöordinate axes to another; and derive from your general result, and also from special investigations, the usual expressions for the following cases:—

(*a*) Rectangular axes turned about $z$ through any angle.

(*b*) Rectangular axes turned into any new position by rotation about a line equally inclined to the three.

(*c*) Rectangular turned to oblique, one of the new axes lying in each of the former cöordinate planes.

2. Point out the distinction between

$$\left(\frac{\alpha+\beta}{\alpha}\right)^2 \text{ and } \frac{(\alpha+\beta)^2}{\alpha^2};$$

and find the value of their difference.

If $$T\beta/\alpha=1, \text{ then } U\frac{\alpha+\beta}{\alpha}=\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}.$$

Shew also that $$\frac{\alpha+\beta}{\alpha-\beta}=\frac{V\alpha\beta}{1+S\alpha\beta},$$

and $$\frac{\alpha-\beta}{\alpha+\beta}=-\frac{V\alpha\beta}{1-S\alpha\beta},$$

provided $\alpha$ and $\beta$ be unit-vectors. If these conditions are not fulfilled, what are the true values?

3. Shew that, whatever quaternion $r$ may be, the expression

$$\alpha r+r\beta,$$

in which $\alpha$ and $\beta$ are any two unit-vectors, is reducible to the form

$$l(\alpha+\beta)+m(\alpha\beta-1),$$

where $l$ and $m$ are scalars.

4. If $T\rho=T\alpha=T\beta=1$, and $S.\alpha\beta\rho=0$, shew by direct transformations that

$$S.U(\rho-\alpha)\,U(\rho-\beta)=\pm\sqrt{\tfrac{1}{2}(1-S\alpha\beta)}.$$

Interpret this theorem geometrically.

5. If $S\alpha\beta = 0$, $T\alpha = T\beta = 1$, shew that

$$(1+\alpha^m)\,\beta = 2\cos\frac{m\pi}{4}\,\alpha^{\frac{m}{2}}\,\beta = 2S\alpha^{\frac{m}{2}}\,.\,\alpha^{\frac{m}{2}}\,\beta.$$

6. Put in its simplest form the equation

$$\rho S\,.\,V\alpha\beta V\beta\gamma V\gamma\alpha = aV.V\gamma\alpha V\alpha\beta + bV.V\alpha\beta V\beta\gamma + cV.V\beta\gamma V\gamma\alpha\,;$$

and shew that $$a = S\,.\,\beta\gamma\rho,\ \&c.$$

7. Shew that any quaternion may in general, in one way only, be expressed as a homogeneous linear function of four given quaternions. Point out the nature of the exceptional cases. Also find the simplest form in which any quaternion may generally be expressed in terms of two given quaternions.

8. Prove the following theorems, and exhibit them as properties of determinants:—

(*a*) $S\,.(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = 2S\,.\,\alpha\beta\gamma$,

(*b*) $S\,.\,V\alpha\beta V\beta\gamma V\gamma\alpha = -(S\,.\,\alpha\beta\gamma)^2$,

(*c*) $$S\,.\,V(\alpha+\beta)(\beta+\gamma)\,V(\beta+\gamma)(\gamma+\alpha)\,V(\gamma+\alpha)(\alpha+\beta) = -4\,(S\,.\,\alpha\beta\gamma)^2,$$

(*d*) $S\,.\,V(V\alpha\beta V\beta\gamma)\,V(V\beta\gamma V\gamma\alpha)\,V(V\gamma\alpha V\alpha\beta) = -(S\,.\,\alpha\beta\gamma)^4$,

(*e*) $S\,.\,\delta\epsilon\zeta = -16\,(S\,.\,\alpha\beta\gamma)^4$,

where
$$\delta = V(V(\alpha+\beta)(\beta+\gamma)\,V(\beta+\gamma)(\gamma+\alpha)),$$
$$\epsilon = V(V(\beta+\gamma)(\gamma+\alpha)\,V(\gamma+\alpha)(\alpha+\beta)),$$
$$\zeta = V(V(\gamma+\alpha)(\alpha+\beta)\,V(\alpha+\beta)(\beta+\gamma)).$$

9. Prove the common formula for the product of two determinants of the third order in the form

$$S\,.\,\alpha\beta\gamma S\,.\,\alpha_1\beta_1\gamma_1 = \begin{vmatrix} S\alpha\alpha_1 & S\beta\alpha_1 & S\gamma\alpha_1 \\ S\alpha\beta_1 & S\beta\beta_1 & S\gamma\beta_1 \\ S\alpha\gamma_1 & S\beta\gamma_1 & S\gamma\gamma_1 \end{vmatrix}.$$

10. Shew that, whatever be the eight vectors involved,

$$\begin{vmatrix} S\alpha\alpha_1 & S\alpha\beta_1 & S\alpha\gamma_1 & S\alpha\delta_1 \\ S\beta\alpha_1 & S\beta\beta_1 & S\beta\gamma_1 & S\beta\delta_1 \\ S\gamma\alpha_1 & S\gamma\beta_1 & S\gamma\gamma_1 & S\gamma\delta_1 \\ S\delta\alpha_1 & S\delta\beta_1 & S\delta\gamma_1 & S\delta\delta_1 \end{vmatrix} = S\,.\,\alpha\beta\gamma S\,.\,\beta_1\gamma_1\delta_1 S\alpha_1(\delta-\delta) = 0.$$

If the single term $S\alpha\alpha_1$ be changed to $S\alpha_0\alpha_1$, the value of the determinant is

$$S\,.\,\beta\gamma\delta S\,.\,\beta_1\gamma_1\delta_1 S\alpha_1(\alpha_0-\alpha).$$

State these as propositions in spherical trigonometry.

Form the corresponding null determinant for any two groups of five quaternions: and give its geometrical interpretation.

11. If, in § 102, $\alpha$, $\beta$, $\gamma$ be three mutually perpendicular vectors, can anything be predicated as to $\alpha_1$, $\beta_1$, $\gamma_1$? If $\alpha$, $\beta$, $\gamma$ be rectangular *unit*-vectors, what of $\alpha_1$, $\beta_1$, $\gamma_1$?

12. If $\alpha$, $\beta$, $\gamma$, $\alpha'$, $\beta'$, $\gamma'$ be two sets of rectangular unit-vectors, shew that

$$S\alpha\alpha' = S\gamma\beta' S\beta\gamma' - S\beta\beta' S\gamma\gamma', \text{ \&c., \&c.}$$

13. The lines bisecting pairs of opposite sides of a quadrilateral (plane or gauche) are perpendicular to each other when the diagonals of the quadrilateral are equal.

14. Shew that

(*a*) $S \,.\, q^2 = 2S^2 q - T^2 q$,

(*b*) $S \,.\, q^3 = S^3 q - 3SqT^2 Vq$,

(*c*) $\alpha^2\beta^2\gamma^2 + S^2 \,.\, \alpha\beta\gamma = V^2 \,.\, \alpha\beta\gamma$,

(*d*) $S(V \,.\, \alpha\beta\gamma V \,.\, \beta\gamma\alpha V \,.\, \gamma\alpha\beta) = 4S\alpha\beta S\beta\gamma S\gamma\alpha S \,.\, \alpha\beta\gamma$,

(*e*) $V \,.\, q^3 = (3S^2 q - T^2 Vq) Vq$,

(*f*) $qUVq^{-1} = -Sq \,.\, UVq + TVq$;

and interpret each as a formula in plane or spherical trigonometry.

15. If $q$ be an undetermined quaternion, what loci are represented by

(*a*) $(q\alpha^{-1})^2 = -a^2$,

(*b*) $(q\alpha^{-1})^4 = a^4$,

(*c*) $S \,.\, (q-\alpha)^2 = a^2$,

where $a$ is any given scalar and $\alpha$ any given vector?

16. If $q$ be any quaternion, shew that the equation

$$Q^2 = q^2$$

is satisfied, not alone by $Q = \pm q$, but also by

$$Q = \pm \sqrt{-1}\,(Sq \,.\, UVq - TVq).$$

(Hamilton, *Lectures*, p. 673.)

17. Wherein consists the difference between the two equations

$$T^2 \frac{\rho}{\alpha} = 1, \text{ and } \left(\frac{\rho}{\alpha}\right)^2 = -1\,?$$

What is the full interpretation of each, $\alpha$ being a given, and $\rho$ an undetermined, vector?

18. Find the *full* consequences of each of the following groups of equations, as regards both the unknown vector $\rho$ and the given vectors $\alpha$, $\beta$, $\gamma$:—

$$(a)\ \begin{array}{l} S.\alpha\beta\rho=0, \\ S.\beta\gamma\rho=0; \end{array} \quad (b)\ \begin{array}{ll} S\alpha\rho & =0, \\ S.\alpha\beta\rho & =0, \\ S\beta\rho & =0; \end{array} \quad (c)\ \begin{array}{ll} S\alpha\rho & =0, \\ S.\alpha\beta\rho & =0, \\ S.\alpha\beta\gamma\rho & =0. \end{array}$$

19. From §§ 74, 110, shew that, if $\epsilon$ be any unit-vector, and $m$ any scalar,

$$\epsilon^m = \cos\frac{m\pi}{2} + \epsilon \sin\frac{m\pi}{2}.$$

Hence shew that if $\alpha$, $\beta$, $\gamma$ be radii drawn to the corners of a triangle on the unit-sphere, whose spherical excess is $m$ right angles,

$$\frac{\alpha+\beta}{\beta+\gamma}\cdot\frac{\gamma+\alpha}{\alpha+\beta}\cdot\frac{\beta+\gamma}{\gamma+\alpha}=\alpha^m.$$

Also that, if $A$, $B$, $C$ be the angles of the triangle, we have

$$\gamma^{\frac{2C}{\pi}}\beta^{\frac{2B}{\pi}}\alpha^{\frac{2A}{\pi}}=-1.$$

20. Shew that for *any* three vectors $\alpha$, $\beta$, $\gamma$ we have

$$(U\alpha\beta)^2+(U\beta\gamma)^2+(U\alpha\gamma)^2+(U.\alpha\beta\gamma)^2+4\,U\alpha\gamma.SU\alpha\beta SU\beta\gamma=-2.$$

(Hamilton, *Elements*, p. 388.)

21. If $a_1$, $a_2$, $a_3$, $x$ be any four scalars, and $\rho_1$, $\rho_2$, $\rho_3$ any three vectors, shew that

$$\begin{aligned}(S.\rho_1\rho_2\rho_3)^2+(\Sigma.a_1V\rho_2\rho_3)^2+x^2(\Sigma V\rho_1\rho_2)^2-x^2(\Sigma.a_1(\rho_2-\rho_3))^2 \\ +2\Pi(x^2+S\rho_1\rho_2+a_1a_2)=2\Pi(x^2+\rho^2)+2\Pi a^2 \\ +\Sigma\{(x^2+a_1^2+\rho_1^2)((V\rho_2\rho_3)^2+2a_2a_3(x^2+S\rho_2\rho_3)-x^2(\rho_2-\rho_3)^2)\};\end{aligned}$$

where $$\Pi a^2=a_1^2a_2^2a_3^2.$$

Verify this formula by a simple process in the particular case

$$a_1=a_2=a_3=x=0.$$

*(Ibid.)*

22. Eliminate $\rho$ from the equations

$$V.\beta\rho\alpha\rho=0,\ S\gamma\rho=0;$$

and state the problem and its solution in a geometrical form.

23. If $p$, $q$, $r$, $s$ be four versors, such that

$$qp = -sr = \alpha,$$

$$rq = -ps = \beta,$$

where $\alpha$ and $\beta$ are unit-vectors; shew that

$$S(V.VsVq\,V.VrVp) = 0.$$

Interpret this as a property of a spherical quadrilateral.

24. Shew that, if $pq$, $rs$, $pr$, and $qs$ be vectors, we have

$$S(V.VpVs\,V.VqVr) = 0.$$

25. If $\alpha$, $\beta$, $\gamma$ be unit-vectors,

$$V\beta\gamma S.\alpha\beta\gamma = -\alpha(1 - S^2\beta\gamma) - \beta(S\alpha\gamma S\beta r + S\alpha\beta)$$
$$- \gamma(S\alpha\beta S\beta\gamma + S\alpha\gamma).$$

26. If $i$, $j$, $k$, $i'$, $j'$, $k'$, be two sets of rectangular unit-vectors, shew that

$$S.Vii'Vjj'Vkk' = (Sij')^2 - (Sji')^2$$
$$= (Sjk')^2 - (Skj')^2 = \&c.,$$

and find the values of the vector of the same product.

27. If $\alpha$, $\beta$, $\gamma$ be a rectangular unit-vector system, shew that, whatever be $\lambda$, $\mu$, $\nu$,

$$\lambda S^2 i\alpha + \mu S^2 j\gamma + \nu S^2 k\beta,$$

$$\lambda S^2 k\gamma + \mu S^2 i\beta + \nu S^2 j\alpha,$$

and

$$\lambda S^2 j\beta + \mu S^2 k\alpha + \nu S^2 i\gamma,$$

are coplanar vectors. What is the connection between this and the result of the preceding example?

# CHAPTER IV.

## DIFFERENTIATION OF QUATERNIONS.

135. In Chapter I. we have already considered as a special case the differentiation of a *vector* function of a scalar independent variable: and it is easy to see at once that a similar process is applicable to a *quaternion* function of a scalar independent variable. The differential, or differential coefficient, thus found, is in general another function of the same scalar variable; and can therefore be differentiated anew by a second, third, &c. application of the same process. And precisely similar remarks apply to partial differentiation of a quaternion function of any number of *scalar* independent variables. In fact, this process is identical with ordinary differentiation.

136. But when we come to differentiate a function of a vector, or of a quaternion, some caution is requisite; there is, in general (except, of course, when the independent variable is a mere scalar), nothing which can be called a differential coefficient; and in fact we require (as already hinted in § 33) to employ a definition of a differential, somewhat different from the ordinary one but, coinciding with it when applied to functions of mere scalar variables.

137. If $r = F(q)$ be a function of a quaternion $q$,

$$dr = dFq = \mathfrak{L}_{\infty} n \left\{F\left(q + \frac{dq}{n}\right) - F(q)\right\},$$

where $n$ is a scalar which is ultimately to be made infinite, is *defined* to be the differential of $r$ or $Fq$.

Here $dq$ may be *any quaternion whatever*, and the right-hand member may be written

$$f(q, dq),$$

where $f$ is a new function, depending on the form of $F$; homogeneous and of the *first* degree in $dq$; but not, in general, capable of being put in the form

$$\mathrm{f}\,(q)\,dq.$$

138. To make more clear these last remarks, we may observe that the function

$$f\,(q,\,dq),$$

thus derived as the differential of $F\,(q)$, is *distributive* with respect to $dq$. That is

$$f\,(q,\,r+s)=f\,(q,\,r)+f\,(q,\,s),$$

$r$ and $s$ being any quaternions.

For $$f\,(q,\,r+s)=\mathcal{L}_\infty\, n\left\{F\left(q+\frac{r+s}{n}\right)-F\,(q)\right\}$$

$$=\mathcal{L}_\infty\, n\left\{F\left(q+\frac{r}{n}+\frac{s}{n}\right)-F\left(q+\frac{s}{n}\right)+F\left(q+\frac{s}{n}\right)-Fq\right\}$$

$$=\mathcal{L}_\infty\, n\left\{F\left(q+\frac{s}{n}+\frac{r}{n}\right)-F\left(q+\frac{s}{n}\right)\right\}+\mathcal{L}_\infty\, n\left\{F\left(q+\frac{s}{n}\right)-Fq\right\}$$

$$=f\,(q,\,r)+f\,(q,\,s).$$

And, as a particular case, it is obvious that if $x$ be any scalar,

$$f\,(q,\,xr)=xf\,(q,\,r).$$

139. And if we define in the same way

$$dF\,(q,\,r,\,s\,\ldots\ldots)$$

as being the value of

$$\mathcal{L}_\infty n\left\{F\left(q+\frac{dq}{n},\,r+\frac{dr}{n},\,s+\frac{ds}{n},\ldots\ldots\right)-F\,(q,\,r,\,s,\,\ldots\ldots)\right\},$$

where $q,\,r,\,s,\ldots\,dq,\,dr,\,ds,\ldots\ldots$ are any quaternions whatever; we shall obviously arrive at a result which may be written

$$f\,(q,\,r,\,s,\ldots dq,\,dr,\,ds,\ldots\ldots),$$

where $f$ is homogeneous and linear in the system of quaternions $dq,\,dr,\,ds,\ldots\ldots$ and distributive with respect to each of them. Thus, in differentiating any power, product, &c. of one or more quaternions, each factor is to be differentiated as if it alone were variable; and the terms corresponding to these are to be added for the complete differential. This differs from the ordinary process of scalar differentiation solely in the fact that, on account of the non-commutative property of quaternion multiplication, each factor must in general be differentiated *in situ*. Thus

$d\,(qr)=dq\,.\,r+qdr$, but *not* generally $=rdq+qdr$.

140. As Examples we take chiefly those which lead to results which will be of constant use to us in succeeding Chapters. Some of the work will be given at full length as an exercise in quaternion transformations.

(1) $(T\rho)^2 = -\rho^2$.

The differential of the left-hand side is simply, since $T\rho$ is a scalar,

$$2T\rho\, dT\rho.$$

That of $\rho^2$ is

$$\mathcal{L}_\infty\, n\left\{\left(\rho + \frac{d\rho}{n}\right)^2 - \rho^2\right\}$$

$$= \mathcal{L}_\infty\, n\left(\frac{2}{n}S\rho d\rho + \frac{(d\rho)^2}{n^2}\right) \text{ (§ 104)}$$

$$= 2S\rho d\rho.$$

Hence

$$T\rho\, dT\rho = -S\rho d\rho,$$

or

$$dT\rho = -S\,.\,U\rho\, d\rho = S\frac{d\rho}{U\rho},$$

or

$$\frac{dT\rho}{T\rho} = S\frac{d\rho}{\rho}.$$

(2) Again, $\rho = T\rho\, U\rho$

$$d\rho = dT\rho\,.\,U\rho + T\rho\, dU\rho,$$

whence

$$\frac{d\rho}{\rho} = \frac{dT\rho}{T\rho} + \frac{dU\rho}{U\rho}$$

$$= S\frac{d\rho}{\rho} + \frac{dU\rho}{U\rho} \text{ by (1).}$$

Hence

$$\frac{dU\rho}{U\rho} = V\frac{d\rho}{\rho}.$$

This may be transformed into $V\dfrac{d\rho\,.\,\rho}{\rho^2}$ or $\dfrac{V\rho d\rho}{T\rho^2}$, &c.

(3) $(Tq)^2 = qKq$

$$2Tq\, dTq = d\,(qKq) = \mathcal{L}_\infty\, n\left[\left(q + \frac{dq}{n}\right)K\left(q + \frac{dq}{n}\right) - qKq\right],$$

$$= \mathcal{L}_\infty\, n\left(\frac{qKdq + dqKq}{n} + \frac{1}{n^2}dqKdq\right),$$

$$= qKdq + dqKq,$$

$$= qKdq + K\,(qKdq) \text{ (§ 55),}$$

$$= 2S\,.\,qKdq = 2S\,.\,dqKq.$$

Hence $$dTq = S \,.\, dqUKq = S \,.\, dqUq^{-1} = TqS\frac{dq}{q},$$

since $$Tq = TKq, \quad \text{and} \quad UKq = Uq^{-1}.$$

[If $q = \rho$, a vector, $Kq = K\rho = -\rho$, and the formula becomes
$$dT\rho = -S \,.\, U\rho\, d\rho, \text{ as in (1).]}$$

Again, $$dq = TqdUq + UqdTq,$$

which gives $$\frac{dq}{q} = \frac{dTq}{Tq} + \frac{dUq}{Uq};$$

whence, as $$S\frac{dq}{q} = \frac{dTq}{Tq},$$

we have $$V\frac{dq}{q} = \frac{dUq}{Uq}.$$

(4) $$\begin{aligned} d(q^2) &= \mathcal{L}_{\infty} n \left\{ \left(q + \frac{dq}{n}\right)^2 - q^2 \right\} \\ &= qdq + dq \,.\, q \\ &= 2S \,.\, qdq + 2Sq \,.\, Vdq + 2Sdq \,.\, Vq. \end{aligned}$$

If $q$ be a vector, as $\rho$, $Sq$ and $Sdq$ vanish, and we have
$$d(\rho^2) = 2S\rho d\rho, \text{ as in (1).}$$

(5) Let $q = r^{\frac{1}{2}}$.

This gives $dr^{\frac{1}{2}} = dq$. But
$$dr = d(q^2) = qdq + dq \,.\, q.$$

This, multiplied *by* $q$ and *into* $Kq$, gives the two equations
$$qdr = q^2dq + qdq \,.\, q,$$

and $$drKq = dq \,.\, Tq^2 + qdq \,.\, Kq.$$

Adding, we have
$$qdr + dr \,.\, Kq = (q^2 + Tq^2 + 2Sq \,.\, q)\, dq = 4Sq \,.\, qdq;$$

whence $dq$, i.e. $dr^{\frac{1}{2}}$, is at once found in terms of $dr$. This process is given by Hamilton, *Lectures*, p. 628. See also § 193 below, and No. 7 of the *Miscellaneous Examples* at the end of this work.

(6) $$\begin{aligned} qq^{-1} &= 1, \\ qdq^{-1} + dq \,.\, q^{-1} &= 0; \\ \therefore \quad dq^{-1} &= -q^{-1}dq \,.\, q^{-1}. \end{aligned}$$

If $q$ is a vector, $= \rho$ suppose,
$$dq^{-1} = d\rho^{-1} = -\rho^{-1}d\rho \,.\, \rho^{-1}$$

$$= \frac{d\rho}{\rho^2} - \frac{2}{\rho} S \frac{d\rho}{\rho}$$

$$= \left(\frac{d\rho}{\rho} - 2S\frac{d\rho}{\rho}\right)\frac{1}{\rho}$$

$$= -K\left(\frac{d\rho}{\rho}\right)\frac{1}{\rho}.$$

(7) $$q = Sq + Vq,$$

$$dq = dSq + dVq.$$

But $$dq = Sdq + Vdq.$$

Comparing, we have

$$dSq = Sdq, \quad dVq = Vdq.$$

Since $Kq = Sq - Vq$, we find by a similar process

$$dKq = Kdq.$$

(8) In the expression $qaq^{-1}$, where $a$ is any constant quaternion, $q$ may be regarded as a mere versor, so that

$$(Tq)^2 = 1 = qKq = qq^{-1}.$$

Thus $$S.dqKq = 0;$$

and hence $$dqq^{-1} = -qdq^{-1},$$

as well as

$$q^{-1}dq = -dq^{-1}q,$$

are vectors. But, if $a = \mathrm{a} + \alpha$, where $\mathrm{a}$ is a scalar, $qaq^{-1} = \mathrm{a}$, i.e. constant, so that we are concerned only with $d(q\alpha q^{-1})$.

Hence $$d(q\alpha q^{-1}) = dq\,\alpha q^{-1} - q\alpha q^{-1}dqq^{-1},$$

$$= dqq^{-1}.q\alpha q^{-1} - q\alpha q^{-1}.dqq^{-1},$$

$$= 2V.dqq^{-1}q\alpha q^{-1} = -2V.qdq^{-1}q\alpha q^{-1}.$$

(9) With the restriction in (8) above we may write

$$q = \cos u + \theta \sin u,$$

where $$T\theta = 1; \quad S\theta d\theta = 0.$$

Hence $$q^{-1} = \cos u - \theta \sin u;$$

$$-q^{-1}dq = dq^{-1}q = \{-(\sin u + \theta\cos u)\,du - d\theta\sin u\}(\cos u + \theta\sin u)$$

$$= -\theta\,du - d\theta\sin u\,(\cos u + \theta\sin u);$$

$$-qdq^{-1} = dq\;q^{-1} = \quad \theta\,du + d\theta\sin u\,(\cos u - \theta\sin u).$$

Both forms are represented as linear functions of the rectangular system of vectors

$$\theta,\ d\theta,\ \theta d\theta.$$

If the plane of $q$ be fixed, $\theta$ is a constant unit vector, and

$$dq\,q^{-1} = -dq^{-1}\,q = \theta du.$$

(10) The equation (belonging to a family of spheres)

$$T\frac{\rho+\alpha}{\rho-\alpha} = e$$

gives $$S\,d\rho\,\{(\rho+\alpha) - e^2(\rho-\alpha)\} = 0\,;$$

or, by elimination of $e$,

$$S\,d\rho\,\{(\rho+\alpha)^{-1} - (\rho-\alpha)^{-1}\} = 0,$$

whose geometrical interpretation gives a well-known theorem.

If we confine our attention to a plane section through the vector $\alpha$, viz.

$$S.\gamma\alpha\rho = 0,\;\; S.\gamma\alpha d\rho = 0\,;\quad \text{or}\quad S\beta\rho = 0,\;\; S\beta d\rho = 0,$$

where $$\beta \,\|\, V\gamma\alpha \,\|\, V\alpha\rho\,;$$

we have

$$d\rho \,\|\, V.\beta\,\{(\rho+\alpha)^{-1} - (\rho-\alpha)^{-1}\}\;\; \text{or}\;\; V.d\rho\,V\beta\,\{(\rho+\alpha)^{-1} - (\rho-\alpha)^{-1}\} = 0.$$

(11) Again, from

$$SU\frac{\rho+\alpha}{\rho-\alpha} = e$$

(which is the equation of the family of *tores* produced by the rotation of a group of circles about their common chord) we have

$$SU.(\rho+\alpha)(\rho-\alpha) = -e.$$

Also this gives $$VU.(\rho+\alpha)(\rho-\alpha) = \beta = \sqrt{1-e^2}\,U.V\alpha\rho.$$

We obtain from the first of these, by differentiation,

$$S\left\{V\frac{d\rho}{\rho+\alpha}.\,U(\rho+\alpha)\;U(\rho-\alpha) + U(\rho+\alpha)\;V\frac{d\rho}{\rho-\alpha}.\,U(\rho-\alpha)\right\} = 0,$$

or $$S.\beta d\rho\,\{(\rho+\alpha)^{-1} - (\rho-\alpha)^{-1}\} = 0.$$

If we consider $\beta$ to be constant, we limit ourselves to a meridian section of the surface (i.e. a circle) and the form of the equation shews that, as $\beta$ is *perpendicular* to the plane of $\alpha$, $\rho$ (and, of course, $d\rho$),

$$V.d\rho\,\{(\rho+\alpha)^{-1} - (\rho-\alpha)^{-1}\} = 0.$$

We leave to the reader the differentiation of the vector form of the equation above.

These results are useful, not only as elementary proofs of geometrical theorems but, as hints on the integration of various simple forms.

(12) As a final instance, take the equation

$$\rho^{-1}\rho'\rho = \alpha,$$

where $\rho'$ stands for $d\rho/ds$, $s$ being the arc of a curve.

By § 38, $\alpha$ is a unit vector, and the expression shews by its form that it belongs to a plane curve. Let $\beta$ be a vector *in* its plane, and perpendicular to $\alpha$. Operate by $S.\beta$ and we get

$$\frac{2\,S\beta\rho}{\rho^2}\,S\rho\rho' - S\beta\rho' = 0,$$

whose integral is

$$\rho^2 - S\beta\rho = 0,$$

the tensor of $\beta$ being the constant of integration.

141. Successive differentiation of course presents no new difficulty.

Thus, we have seen that

$$d\,(q^2) = dq\,.\,q + qdq.$$

Differentiating again, we have

$$d^2\,(q^2) = d^2q\,.\,q + 2\,(dq)^2 + qd^2q,$$

and so on for higher orders.

If $q$ be a vector, as $\rho$, we have, § 140 (1),

$$d\,(\rho^2) = 2S\rho d\rho.$$

Hence $d^2\,(\rho^2) = 2\,(d\rho)^2 + 2S\rho d^2\rho$, and so on.

Similarly $$d^2U\rho = -\,d\left(\frac{U\rho}{T\rho^2}\,V\rho d\rho\right).$$

But $$d\,\frac{1}{T\rho^2} = -\,\frac{2dT\rho}{T\rho^3} = \frac{2S\rho d\rho}{T\rho^4},$$

and $$d\,.\,V\rho d\rho = V\,.\,\rho d^2\rho.$$

Hence $$d^2U\rho = \frac{U\rho}{T\rho^4}\,(V\rho d\rho)^2 - \frac{U\rho V\rho d^2\rho}{T\rho^2} - \frac{2\,U\rho V\rho d\rho S\rho d\rho}{T\rho^4}$$

$$= \frac{U\rho}{T\rho^4}\,\{(V\rho d\rho)^2 + \rho^2\,V\rho d^2\rho - 2\,V\rho d\rho S\rho d\rho\}^*.$$

142. If the first differential of $q$ be considered as a *constant* quaternion, we have, of course,

$$d^2q = 0, \qquad d^3q = 0, \text{ \&c.}$$

and the preceding formulæ become considerably simplified.

---

* This may be farther simplified; but it may be well to caution the student that we cannot, for such a purpose, write the above expression as

$$\frac{U\rho}{T\rho^4}\,V.\rho\{d\rho V\rho d\rho + d^2\rho\,.\,\rho^2 - 2d\rho S\rho d\rho\}.$$

Hamilton has shewn that in this case *Taylor's Theorem* admits of an easy extension to quaternions. That is, we may write

$$f(q+xdq)=f(q)+xdf(q)+\frac{x^2}{1.2}d^2f(q)+\ldots\ldots$$

if $d^2q=0$; subject, of course, to particular exceptions and limitations as in the ordinary applications to functions of scalar variables. Thus, let

$$f(q)=q^3, \text{ and we have}$$
$$df(q)=q^2dq+qdq.q+dq.q^2,$$
$$d^2f(q)=2dq.qdq+2q(dq)^2+2(dq)^2q,$$
$$d^3f(q)=6(dq)^3,$$

and it is easy to verify by multiplication that we have rigorously

$$(q+xdq)^3=q^3+x(q^2dq+qdq.q+dq.q^2)+$$
$$x^2(dq.qdq+q(dq)^2+(dq)^2q)+x^3(dq)^3;$$

which is the value given by the application of the above form of Taylor's Theorem.

As we shall not have occasion to employ this theorem, and as the demonstrations which have been found are all too laborious for an elementary treatise, we refer the reader to Hamilton's works, where he will find several of them.

143. To differentiate a function of a function of a quaternion we proceed as with scalar variables, attending to the peculiarities already pointed out.

144. A case of considerable importance in geometrical and physical applications of quaternions is the differentiation of a scalar function of $\rho$, the vector of any point in space.

Let $$F(\rho)=C,$$

where $F$ is a scalar function and $C$ an arbitrary constant, be the equation of a series of surfaces. Its differential,

$$f(\rho, d\rho)=0,$$

is, of course, a scalar function: and, being homogeneous and linear in $d\rho$, § 137, may be thus written,

$$S\nu d\rho=0,$$

where $\nu$ is a vector, in general a function of $\rho$.

This vector, $\nu$, is shewn, by the last written equation, to have the direction of the *normal* to the given surface at the extremity of $\rho$. It is, in fact, perpendicular to every tangent line $d\rho$; §§ 36, 98.

145. This leads us directly to one of the most remarkable operators peculiar to the Quaternion Calculus; viz.

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz} \quad \text{.....................(1)},$$

to whose elementary properties we will devote the remainder of the chapter. The above definition is that originally given by Hamilton, before the calculus had even partially thrown off its early Cartesian trammels. Since $i$, $j$, $k$ stand for *any* system of rectangular unit vectors, while $x$, $y$, $z$ are Cartesian co-ordinates referred to these as axes, it is implied in (1) that $\nabla$ is an *Invariant*. This will presently be justified. Meanwhile it is easy to see that if $\rho$ be the vector of any point in space, so that

$$\rho = ix + jy + kz,$$

we have
$$\nabla\rho = -3 \quad \text{.................................(2)},$$

$$\nabla T\rho = \nabla\sqrt{x^2+y^2+z^2} = \frac{ix+jy+kz}{\sqrt{x^2+y^2+z^2}} = \frac{\rho}{T\rho} = U\rho \quad \text{...(3)},$$

$$\nabla(T\rho)^n = n(T\rho)^{n-1}\nabla T\rho = n(T\rho)^{n-2}\rho \text{...........(4)},$$

of which the most important case is

$$\nabla\frac{1}{T\rho} = -\frac{\rho}{(T\rho)^3} = -\frac{U\rho}{T\rho^2} \text{....................(5)}.$$

A second application gives

$$\nabla^2\frac{1}{T\rho} = -\frac{\nabla\rho}{T\rho^3} - \nabla\frac{1}{T\rho^3}\cdot\rho = 0 \quad \text{..............(6)}.$$

Again

$$\nabla\rho = -3 = \nabla T\rho\,.\,U\rho + T\rho\,.\,\nabla U\rho = -1 + T\rho\,.\,\nabla U\rho,$$

so that
$$\nabla U\rho = -\frac{2}{T\rho} \text{..........................(7)}.$$

By the definition (1) we see that

$$\nabla^2 = -\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\} \text{..............(8)},$$

the negative of what has been called *Laplace's Operator*. Thus (6) is merely a special case of Laplace's equation for the potential in free space.

Again we see by (2), $\alpha$ being any constant vector,

$$S\alpha\nabla\,.\,\rho - \nabla S\alpha\rho = V\,.\,\alpha V\nabla\rho = 0,$$

from which

$$\nabla V\alpha\rho + V\alpha\nabla\,.\,\rho = (S\alpha\nabla\,.\,\rho - \alpha S\nabla\rho) + (\alpha S\nabla\rho - \nabla S\alpha\rho) = 0.$$

[The student should note here that, in expanding the terms of the vector function on the left by the formula (1) of § 90, the partial terms are written so that $\nabla$ is *always* to the left of (though not necessarily contiguous to) its subject, $\rho$.]

146. By the help of these elementary results, of which (3) and (7) are specially noteworthy, we easily find the effect of $\nabla$ upon more complex functions.

For instance, taking different modes of operating, we have with

$$\alpha = ia + jb + kc$$

$$S\alpha\nabla \, . \, \rho = \nabla S\alpha\rho = -\nabla(ax + by + cz) = -(ia + jb + kc) = -\alpha \ldots (1),$$

or thus
$$\nabla S\alpha\rho = iS\alpha i + jS\alpha j + kS\alpha k = -\alpha\,;$$

while
$$-V\alpha\nabla \, . \, \rho = \nabla V\alpha\rho = -\nabla V\rho\alpha = -\nabla(\rho\alpha - S\rho\alpha)$$
$$= 3\alpha - \alpha = 2\alpha \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

or
$$\nabla V\alpha\rho = iV\alpha i + jV\alpha j + kV\alpha k = 2\alpha.$$

From the latter of these we have

$$\nabla \frac{V\alpha\rho}{T\rho^3} = \frac{2\alpha}{T\rho^3} - \frac{3\rho V\alpha\rho}{T\rho^5} = -\frac{2\alpha\rho^2 + 3\rho V\alpha\rho}{T\rho^5} = \frac{\alpha\rho^2 - 3\rho S\alpha\rho}{T\rho^5} \ldots (3),$$

[where note that the first of these values is obtained thus,

$$\nabla \frac{V\alpha\rho}{T\rho^3} = \frac{\nabla V\alpha\rho}{T\rho^3} + \nabla \frac{1}{T\rho^3} \, . \, V\alpha\rho.$$

The order is of vital importance.]

This, in its turn, gives

$$S \, . \, \delta\rho\nabla \frac{V\alpha\rho}{T\rho^3} = -\frac{S\alpha\delta\rho}{T\rho^3} - \frac{3S\alpha\rho S\rho\delta\rho}{T\rho^5} = -\delta \frac{S\alpha\rho}{T\rho^3} \ldots\ldots (4),$$

where $\delta$ is a symbol of variation. This is a result of great physical importance, especially in electro-dynamics. We may alter the right-hand member (by § 145, (5)) so as to write the whole in the form

$$S \, . \, \delta\rho\nabla \frac{V\alpha\rho}{T\rho^3} = \delta S\alpha\nabla \, . \, \frac{1}{T\rho} = S\alpha\nabla \, . \, \delta \frac{1}{T\rho} \ldots\ldots\ldots (4').$$

And it is easy to see that $S$ may be substituted for $V$ in the left-hand member. [The reason for this may be traced in the result of § 145 (6).]

As an addition to these examples, note that (by (2) of § 148, below)

$$S\delta\rho\nabla \, . \, \frac{V\alpha\rho}{T\rho^3} = -\delta \frac{V\alpha\rho}{T\rho^3},$$

which may be contrasted with (4) above. The altered position of the point produces a complete change in the meaning of the left-hand member.

Finally, we see that

$$\nabla \Sigma \alpha S\beta\rho = - \Sigma \beta\alpha \quad \text{.......................(5)},$$

a result which will be found useful in next Chapter.

147. Still more important are the results obtained from the operator $\nabla$ when it is applied to

$$\sigma = i\xi + j\eta + k\zeta \quad \text{.......................(1)},$$

and functions of this vector. (Here $\xi$, $\eta$, $\zeta$ are functions of $x$, $y$, $z$, so that $\sigma$ is a vector whose value is definite for each point of space.)

We have at once

$$\nabla\sigma = -\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) + i\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) + j\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) + k\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right) \quad \text{...............(2)}.$$

Those who are acquainted with mathematical physics will recognize at a glance the importance of this expression. For, if $\sigma$ denote the displacement (or the velocity) of a point originally situated at $\rho$, it is clear that

$$S\nabla\sigma = -\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) \quad \text{...............(3)},$$

represents the consequent condensation of the group of points (say particles of a fluid) originally in the neighbourhood of $\rho$, while

$$V\nabla\sigma = i\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right) + j\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right) + k\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right) \quad \text{...(4)},$$

represents double the (vector) axis of rotation of the same group.

Other, and more purely quaternion, methods will be employed later, to deduce these results afresh, and to develop their applications. They are introduced here in their semi-Cartesian form merely to shew the importance of the operator $\nabla$.

148. Let us recur to the equation of § 144, viz.

$$F(\rho) = C \quad \text{...............................(1)}.$$

Ordinary complete differentiation gives

$$dF = \frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dx,$$

or, what is obviously the same,

$$dF = -Sd\rho\nabla F \text{ .......................}(2),$$

which we may write, if we please, as

$$-Sd\rho\nabla \,.\, F.$$

*Here* the point is obviously unnecessary, but we shall soon come to cases in which it, or some equivalent, is indispensable.

Thus it appears that the operator

$$-Sd\rho\nabla$$

is equivalent to total differentiation as involved in the passage from $\rho$ to $\rho + d\rho$. Hence, of course, as in § 144

$$dF(\rho) = 0 = S\nu d\rho = -Sd\rho\nabla F,$$

and thus (as $d\rho$ may have any of an infinite number of values)

$$\nu = -\nabla F \text{ ..........................}(3).$$

If we pass from one surface of the series (1) to a consecutive one by the vector $\delta\rho$, we have

$$-S\nu\delta\rho = \delta C.$$

Hence $-\nu^{-1}\delta C$ is a value of $\delta\rho$; so that *the tensor of $\nu$ is, at every point, inversely as the normal distance between two consecutive surfaces of the series.*

Thus, if (1) be the equation of a series of equipotential surfaces, $\nu$, as given by (3), represents the vector force at the point $\rho$; if (1) be a set of isothermals, $\nu$ (multiplied by the conductivity, a scalar) is the flux of heat, &c.

149. We may extend the result (2) of § 148 to vector functions by multiplying both sides into a constant vector, $\alpha$, and adding three such expressions together. Thus if

$$\sigma = \alpha F + \beta F_1 + \gamma F_2,$$

we obtain at once

$$d\sigma = -S(d\rho\nabla)\sigma = -Sd\rho\nabla \,.\, \sigma \text{ ...............}(4).$$

But *here* the brackets, or the point, should (at first, at least) be employed; otherwise we might confound the expression with

$$d\sigma = -S \,.\, d\rho\nabla\sigma$$

which, as equating a vector to a scalar, is an absurdity (unless both sides vanish). See again, § 148.

Finally, from (2) and (4), we have for any quaternion

$$dq = -Sd\rho\nabla \,.\, q \text{ .......................}(5).$$

The student's attention is particularly called to the simple processes we have adopted in obtaining (4) and (5) from (2) of § 148; because, in later chapters, other and more complex results obtained by the same processes will frequently be taken for granted; especially when other operators than $S(d\rho\nabla)$ are employed. The precautions necessary in such matters are two-fold, (*a*) *the operator must never be placed anywhere after the operand;* (*b*) *its commutative or non-commutative character must be carefully kept in view.*

## EXAMPLES TO CHAPTER IV.

1. Shew that

(*a*) $d\,.\,SUq = S\,.\,UqV\dfrac{dq}{q} = -S\dfrac{dq}{qUVq}\,TVUq,$

(*b*) $d\,.\,VUq = V\,.\,Uq^{-1}V(dq\,.\,q^{-1}),$

(*c*) $d\,.\,TVUq = S\dfrac{dUq}{UVq} = S\dfrac{dq}{qUVq}\,SUq,$

(*d*) $d\,.\,\alpha^x = \dfrac{\pi}{2}\,\alpha^{x+1}dx,$

(*e*) $d^2\,.\,Tq = \{S^2\,.\,dqq^{-1} - S\,.\,(dqq^{-1})^2\}\,Tq = -Tq\,V^2\dfrac{dq}{q}.$

2. If $F\rho = \Sigma\,.\,S\alpha\rho S\beta\rho + \tfrac{1}{2}g\rho^2$

give $dF\rho = S\nu d\rho,$

shew that $\nu = \Sigma V\,.\,\alpha\rho\beta + (g + \Sigma S\alpha\beta)\,\rho.$

3. Find the maximum and minimum values of $T\rho$, when

(*a*) $(\rho - \alpha)^2 = -a^2;$

(*b*) $\left.\begin{array}{r}(\rho - \alpha)^2 = -a^2,\\ S\beta\rho = 0;\end{array}\right\}$

(*c*) $\rho^2 - S\alpha\rho S\beta\rho = -a^2;$

(*d*) $\left.\begin{array}{r}\rho^2 - S\alpha\rho S\beta\rho = -a^2,\\ S\gamma\rho = 0.\end{array}\right\}$

Point out the differences, geometrical and analytical, between (*a*), (*c*) on the one hand, and (*b*), (*d*) on the other.

State each of the problems in words.

4. With $\nabla$ as in § 145, shew that

$$S\nabla(q\alpha q^{-1}) = 2S.\nabla q\alpha q^{-1} = 2S.\alpha q^{-1}\nabla q.$$

$$V\nabla(q\alpha q^{-1}) = 2q\alpha q^{-1}S.\nabla q q^{-1} - 2S(q\alpha q^{-1}\nabla)q.q^{-1}$$

where $q$ is a function of $\rho$, and $\alpha$ any constant vector.

5. Shew that, if $\alpha$, $\beta$, $\gamma$ be a constant rectangular unit-vector system,

$$q\alpha q^{-1}d.q\alpha q^{-1} + q\beta q^{-1}d.q\beta q^{-1} + q\gamma q^{-1}d.q\beta q^{-1} = 4dqq^{-1}.$$

6. Integrate the differential equations :—

(a) $\dot{\rho} + \alpha S\beta\rho = \gamma$,

(b) $\dot{q} + aq = b$,

(c) $\dot{\theta} = V\alpha\theta$. *Proc. R. S. E.* 1870.

7. Shew that

(a) $\int V\alpha d\rho S\beta\rho = \frac{1}{2}V.\alpha(\rho S\beta\rho + V.\beta\int V\rho d\rho)$.

(b) $\int d\rho V\alpha\rho = \frac{1}{2}(\rho V\alpha\rho - V.\alpha\int V\rho d\rho) + S.\alpha\int V\rho d\rho$.

(c) $\int V.V\alpha d\rho V\beta\rho = \frac{1}{2}(\alpha S.\beta\int V\rho d\rho + \beta S.\alpha\int V\rho d\rho - \rho S.\alpha\beta\rho)$,

(d) $\int S.V\alpha d\rho V\beta\rho = \frac{1}{2}(S\alpha\rho S\beta\rho - \rho^2 S\alpha\beta - S.\alpha\beta\int V\rho d\rho)$.

When these integrals are taken round a closed plane curve we have

$$\int V\rho d\rho = 2A\gamma,$$

where $A$ is the area, and $\gamma$ a unit vector perpendicular to its plane. In this case

$$\int d\rho V\alpha\rho = AV\gamma\alpha + 2AS\gamma\alpha,$$

$$\int V.V\alpha d\rho V\beta\rho = A(\alpha S\beta\gamma + \beta S\alpha\gamma),$$

$$\int S.V\alpha d\rho V\beta\rho = AS.\alpha\beta\gamma.$$

8. State, in words, the geometrical theorems involved in the equations of § 140, (10), (11), (12).

9. Shew (by means of § 91) that

$$\nabla_1 S.\sigma_1\nabla\sigma = S.\nabla\sigma\nabla_1.\sigma_1,$$

where $\nabla$, $\nabla_1$, operate respectively on $\sigma$, $\sigma_1$; but, *after* the operations are performed, we put

$$\nabla_1 = \nabla, \ \sigma_1 = \sigma.$$

# CHAPTER V.

## THE SOLUTION OF EQUATIONS OF THE FIRST DEGREE.

150. We have seen that the differentiation of any function whatever of a quaternion, $q$, leads to an equation of the form

$$dr = f(q, dq),$$

where $f$ is linear and homogeneous in $dq$. To complete the process of differentiation, we must have the means of solving this equation, so as to be able to exhibit directly the value of $dq$.

This general equation is not of so much practical importance as the particular case in which $dq$ is a vector; and, besides, as we proceed to shew, the solution of the general question may easily be made to depend upon that of the particular case; so that we shall commence with the latter.

The most general expression for the function $f$ is easily seen to be

$$dr = f(q, dq) = \Sigma V . a dq b + S . c dq,$$

where $a$, $b$, and $c$ may be any quaternion functions of $q$ whatever, including even scalar constants. Every possible term of a linear and homogeneous function is reducible to this form, as the reader may see at once by writing down all the forms he can devise.

Taking the scalars of both sides, we have

$$Sdr = S . c dq = Sdq Sc + S . Vdq Vc.$$

But we have also, by taking the vector parts,

$$Vdr = \Sigma V . a dq b = Sdq . \Sigma Vab + \Sigma V . a (Vdq) b.$$

Eliminating $Sdq$ between the equations for $Sdr$ and $Vdr$ it is obvious that a linear and vector expression in $Vdq$ will remain. Such an expression, so far as it contains $Vdq$, may always be reduced to the form of a sum of terms of the type $\alpha S . \beta Vdq$, by the help of formulæ like those in §§ 90, 91. Solving this, we have $Vdq$, and $Sdq$ is then found from the preceding equation.

151. The problem may now be stated thus.

Find the value of $\rho$ from the equation

$$\alpha S\beta\rho + \alpha_1 S\beta_1\rho + \ldots = \Sigma . \alpha S\beta\rho = \gamma,$$

where $\alpha$, $\beta$, $\alpha_1$, $\beta_1$,...$\gamma$ are given vectors.

The most general form of the left-hand member requires but three distinct or independent terms. These, however, in consequence of the form of the expression, involve scalar constants only; since the whole can obviously be reduced to terms of the forms $AiSi\rho$, $BiSj\rho$, $CjSj\rho$, &c. and there are only nine such forms. In fact we may write the most general form either as

$$iS\alpha\rho + jS\beta\rho + kS\gamma\rho,$$

or as

$$\alpha_1 Si\rho + \beta_1 Sj\rho + \gamma_1 Sk\rho,$$

according as we arrange it by the vector, or by the scalar, factors of the several terms. But the form

$$\alpha S\beta\rho + \alpha_1 S\beta_1\rho + \alpha_2 S\gamma_2\rho,$$

is that which, as committing us to no special system of vectors of reference, is most convenient for a discussion of its properties.

If we write, with Hamilton,

$$\phi\rho = \Sigma . \alpha S\beta\rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the given equation may be written

$$\phi\rho = \gamma,$$

or

$$\rho = \phi^{-1}\gamma,$$

and the object of our investigation is to find the value of the inverse function $\phi^{-1}$.

It is important to remark that the definition (1) shews $\phi$ to be distributive, so that

$$\phi(\rho + \sigma) = \phi\rho + \phi\sigma.$$

A particular case of this is

$$\phi(x\rho) = x\phi\rho,$$

where $x$ is a scalar.

Also, by the statement above, it is clear that $\phi$, in its most general form, essentially involves nine independent scalars.

152. We have seen that any vector whatever may be expressed linearly in terms of any three non-coplanar vectors. Hence, we should expect *à priori* that a vector such as $\phi\phi\phi\rho$, or $\phi^3\rho$, for instance, should be capable of expression in terms of $\rho$, $\phi\rho$, and $\phi^2\rho$.

[This is, of course, on the supposition that $\rho$, $\phi\rho$, and $\phi^2\rho$ are not generally coplanar. But it may easily be seen to extend to that case also. For if these vectors be generally coplanar, so are $\phi\rho$, $\phi^2\rho$, and $\phi^3\rho$, since they may be written $\sigma$, $\phi\sigma$, and $\phi^2\sigma$. And thus, of course, $\phi^3\rho$ can be expressed as above. If in a particular case, we should have, for some *definite* vector $\rho$, $\phi\rho = g\rho$ where $g$ is a scalar, we shall obviously have $\phi^2\rho = g^2\rho$ and $\phi^3\rho = g^3\rho$, so that the equation will still subsist. And a similar explanation holds for the particular case when, for some *definite* value of $\rho$, the three vectors $\rho$, $\phi\rho$, $\phi^2\rho$ are coplanar. For then we have an equation of the form

$$\phi^2\rho = A\rho + B\phi\rho,$$

which gives

$$\begin{aligned}\phi^3\rho &= A\phi\rho + B\phi^2\rho \\ &= AB\rho + (A + B^2)\,\phi\rho,\end{aligned}$$

so that $\phi^3\rho$ is in the same plane.]

If, then, we write

$$-\phi^3\rho = x\rho + y\phi\rho + z\phi^2\rho \text{ ......................(1)},$$

and bear in mind the distributive character of the operator $\phi$, *it is evident* (if only *ex absurdo*) that $x$, $y$, $z$ are quantities independent of the vector $\rho$.

[The words above, "it is evident," have been objected to by more than one correspondent. But, on full consideration, I not only leave them where they are, but put them in Italics. For they are, of course, addressed to the reader only; and it is to be presumed that, before he reaches them, he has mastered the contents of at least the more important previous sections which bear on this question, such as §§ 23, 151. If, with these sections in his mind, and a *homogeneous* linear equation such as (1) before him, he does not see the "evidence," he has begun the study of Quaternions too soon. A formal demonstration, giving the values of $x$, $y$, $z$, will however be found in §§ 156—9 below.]

If any three vectors, as $i$, $j$, $k$, be substituted for $\rho$, they will in general enable us to assign the values of the three coefficients on the right side of the equation, and the solution of the problem of § 151 is complete. For by putting $\phi^{-1}\rho$ for $\rho$ and transposing, the equation becomes

$$-x\phi^{-1}\rho = y\rho + z\phi\rho + \phi^2\rho\,;$$

that is, the unknown inverse function is expressed in terms of direct operations. Should $x$ vanish, while $y$ remains finite, we must substitute $\phi^{-2}\rho$ for $\rho$, and have

$$-y\phi^{-1}\rho = z\rho + \phi\rho\,;$$

and if $x$ and $y$ both vanish

$$-z\phi^{-1}\rho = \rho.$$

[We may remark here that it is in general possible to determine $x, y, z$ by putting *one* known vector for $\rho$ in (1). The circumstances in which some particular vector does not suffice will be clear from the theory to be given below.]

153. To illustrate this process by a simple example we shall take the very important case in which $\phi$ belongs to a *central* surface of the second order; suppose an ellipsoid; in which case it will be shewn (in Chap. IX.) that we may write

$$\phi\rho = -a^2iSi\rho - b^2jSj\rho - c^2kSk\rho,$$

where $i, j, k$ are parallel to the principal diameters, and the semi-lengths of these are $1/a$, $1/b$, $1/c$.

Here we have

$$\begin{aligned} \phi i &= a^2 i, & \phi^2 i &= a^4 i, & \phi^3 i &= a^6 i, \\ \phi j &= b^2 j, & \phi^2 j &= b^4 j, & \phi^3 j &= b^6 j, \\ \phi k &= c^2 k, & \phi^2 k &= c^4 k, & \phi^3 k &= c^6 k. \end{aligned}$$

Hence, putting separately $i, j, k$ for $\rho$ in the equation (1) of last section, we have

$$\begin{aligned} -a^6 &= x + ya^2 + za^4, \\ -b^6 &= x + yb^2 + zb^4, \\ -c^6 &= x + yc^2 + zc^4. \end{aligned}$$

Hence $a^2$, $b^2$, $c^2$ are the roots of the cubic

$$\xi^3 + z\xi^2 + y\xi + x = 0,$$

which involves the conditions

$$\begin{aligned} z &= -(a^2 + b^2 + c^2), \\ y &= a^2b^2 + b^2c^2 + c^2a^2, \\ x &= -a^2b^2c^2. \end{aligned}$$

Thus, with the above value of $\phi$, we have

$$\phi^3\rho = a^2b^2c^2\rho - (a^2b^2 + b^2c^2 + c^2a^2)\phi\rho + (a^2 + b^2 + c^2)\phi^2\rho.$$

154. Putting $\phi^{-1}\sigma$ in place of $\rho$ (which is *any* vector whatever) and changing the order of the terms, we have the desired inversion of the function $\phi$ in the form

$$a^2b^2c^2\phi^{-1}\sigma = (a^2b^2 + b^2c^2 + c^2a^2)\sigma - (a^2 + b^2 + c^2)\phi\sigma + \phi^2\sigma,$$

where the inverse function is expressed in terms of the direct function. For this particular case the solution we have given is complete, and satisfactory; and it has the advantage of preparing the reader to expect a similar form of solution in more complex cases.

155. It may also be useful as a preparation for what follows, if we put the equation of § 153 in the form

$$\begin{aligned}0=\Phi(\rho)&=\phi^3\rho-(a^2+b^2+c^2)\,\phi^2\rho+(a^2b^2+b^2c^2+c^2a^2)\,\phi\rho-a^2b^2c^2\rho\\&=\{\phi^3-(a^2+b^2+c^2)\,\phi^2+(a^2b^2+b^2c^2+c^2a^2)\,\phi-a^2b^2c^2\}\,\rho\\&=\{(\phi-a^2)\,(\phi-b^2)\,(\phi-c^2)\}\,\rho\ \ldots\ldots\ldots\ldots\ldots\ldots(2).\end{aligned}$$

This last transformation is permitted because (§ 151) $\phi$ is commutative with scalars like $a^2$, i.e. $\phi\,(a^2\rho)=a^2\phi\rho$. The explanation of its meaning must, however, be deferred to a later section. (§ 177.)

Here we remark that the equation

$$V.\rho\phi\rho=0,\quad\text{or}\quad\phi\rho=g\rho,$$

where $\phi$ is as in § 153, and $g$ is some undetermined scalar, is satisfied, not merely by every vector of null-length, but by the definite system of three rectangular vectors $Ai$, $Bj$, $Ck$ whatever be their tensors, the corresponding particular values of $g$ being $a^2$, $b^2$, $c^2$.

156. We now give Hamilton's admirable investigation.

The most general form of a linear and vector function of a vector may of course be written as

$$\phi\rho=\Sigma V.q\rho r,$$

where $q$ and $r$ are any constant quaternions, either or both of which may degrade to a scalar or a vector.

Hence, operating by $S.\sigma$ where $\sigma$ is any vector whatever,

$$S\sigma\phi\rho=S\sigma\Sigma V.q\rho r=S\rho\Sigma V.r\sigma q=S\rho\phi'\sigma\ldots\ldots\ldots\ldots(1),$$

if we agree to write $\quad\phi'\sigma=\Sigma V.r\sigma q,$

and remember the proposition of § 88. The functions $\phi$ and $\phi'$ are thus *conjugate* to one another, and on this property the whole investigation depends.

157. Let $\lambda$, $\mu$ be any two vectors, such that

$$\phi\rho=V\lambda\mu.$$

Operating by $S.\lambda$ and $S.\mu$ we have

$$S\lambda\phi\rho=0,\quad S\mu\phi\rho=0.$$

But, introducing the conjugate function $\phi'$, these become

$$S\rho\phi'\lambda = 0, \quad S\rho\phi'\mu = 0,$$

and give $\rho$ in the form $\quad m\rho = V\phi'\lambda\phi'\mu,$

where $m$ is a scalar which, as we shall presently see, is independent of $\lambda$, $\mu$, and $\rho$.

But our original assumption gives

$$\rho = \phi^{-1}V\lambda\mu;$$

hence we have $\quad m\phi^{-1}V\lambda\mu = V\phi'\lambda\phi'\mu \quad \text{..................... (2)},$

and the problem of inverting $\phi$ is solved.

It remains to find the value of the constant $m$, and to express the vector

$$V\phi'\lambda\phi'\mu$$

as a function of $V\lambda\mu$.

158. To find the value of $m$, we may operate on (2) by $S.\phi'\nu$, where $\nu$ is any vector not coplanar with $\lambda$ and $\mu$, and we get

$$mS.\phi'\nu\phi^{-1}V\lambda\mu = mS.\nu\phi\phi^{-1}V\lambda\mu \quad \text{(by (1) of § 156)}$$

$$= mS.\lambda\mu\nu = S.\phi'\lambda\phi'\mu\phi'\nu, \text{ or}$$

$$m = \frac{S.\phi'\lambda\phi'\mu\phi'\nu}{S.\lambda\mu\nu} \quad \text{..................... (3)}.$$

[That this quantity is independent of the particular vectors $\lambda$, $\mu$, $\nu$ is evident from the fact that if

$$\lambda' = x\lambda + y\mu + z\nu, \quad \mu' = x_1\lambda + y_1\mu + z_1\nu, \quad \text{and} \quad \nu' = x_2\lambda + y_2\mu + z_2\nu$$

be any other three vectors (which is possible since $\lambda$, $\mu$, $\nu$ are not coplanar), we have

$$\phi'\lambda' = x\phi'\lambda + y\phi'\mu + z\phi'\nu, \text{ \&c., \&c.};$$

from which we deduce

$$S.\phi'\lambda'\phi'\mu'\phi'\nu' = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} S.\phi'\lambda\phi'\mu\phi'\nu,$$

and

$$S.\lambda'\mu'\nu' = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} S.\lambda\mu\nu,$$

so that the numerator and denominator of the fraction which expresses $m$ are altered in the same ratio. Each of these quantities is in fact an *Invariant*, and the numerical multiplier is the same for both when we pass from any one set of three vectors to another.

A still simpler proof is obtained at once by writing $\lambda + x\mu$ for $\lambda$ in (3), and noticing that neither numerator nor denominator is altered.]

159. We have next to express

$$V\phi'\lambda\phi'\mu$$

as a function of $V\lambda\mu$. For this purpose let us change $\phi$ to $\phi - g$, where $g$ is any scalar. It is evident that $\phi'$ becomes $\phi' - g$, and our equation (2) becomes

$$\begin{aligned} m_g (\phi - g)^{-1} V\lambda\mu &= V(\phi' - g)\lambda(\phi' - g)\mu; \\ &= V\phi'\lambda\phi'\mu - gV(\phi'\lambda\mu + \lambda\phi'\mu) + gV^2\lambda\mu, \\ &= (m\phi^{-1} - g\chi + g^2)V\lambda\mu, \text{ suppose.} \end{aligned}$$

In this equation (see (3) above)

$$\begin{aligned} m_g &= \frac{S.(\phi' - g)\lambda(\phi' - g)\mu(\phi' - g)\nu}{S.\lambda\mu\nu} \\ &= m - m_1 g + m_2 g^2 - g^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4) \end{aligned}$$

is what $m$ becomes when $\phi$ is changed into $\phi - g$; $m_1$ and $m_2$ being two new scalar constants whose values are

$$m_1 = \frac{S.(\lambda\phi'\mu\phi'\nu + \mu\phi'\nu\phi'\lambda + \nu\phi'\lambda\phi'\mu)}{S.\lambda\mu\nu},$$

$$m_2 = \frac{S.(\lambda\mu\phi'\nu + \mu\nu\phi'\lambda + \nu\lambda\phi'\mu)}{S.\lambda\mu\nu}.$$

If, in these expressions, we put $\lambda + x\mu$ for $\lambda$, we find that the terms in $x$ vanish identically; so that they also are invariants.

Substituting for $m_g$, and equating the coefficients of the various powers of $g$ after operating on both sides by $\phi - g$, we have two identities and the following two equations,

$$m_2 = \phi + \chi,$$
$$m_1 = \phi\chi + m\phi^{-1}.$$

[The first determines $\chi$, and shews that we were justified in treating $V(\phi'\lambda\mu + \lambda\phi'\mu)$ as a linear and vector function of $V.\lambda\mu$. The result might have been also obtained thus,

$$\begin{aligned} S.\lambda\chi V\lambda\mu &= S.\lambda\phi'\lambda\mu = -S.\lambda\mu\phi'\lambda = -S.\lambda\phi V\lambda\mu, \\ S.\mu\chi V\lambda\mu &= S.\mu\lambda\phi'\mu = -S.\mu\phi V\lambda\mu, \\ S.\nu\chi V\lambda\mu &= S.(\nu\phi'\lambda\mu + \nu\lambda\phi'\mu) \\ &= m_2 S\lambda\mu\nu - S.\lambda\mu\phi'\nu \\ &= S.\nu(m_2 V\lambda\mu - \phi V\lambda\mu); \end{aligned}$$

and all three (the utmost generality) are satisfied by

$$\chi = m_2 - \phi.]$$

160. Eliminating $\chi$ from these equations we find

$$m_1 = \phi\,(m_2 - \phi) + m\phi^{-1},$$

or

$$m\phi^{-1} = m_1 - m_2\,\phi + \phi^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

which contains the complete solution of linear and vector equations.

161. More to satisfy the student as to the validity of the above investigation, about whose logic he may at first feel some difficulties, than to obtain easy solutions, we take a few very simple examples to begin with: we treat them with all desirable prolixity, as useful practice in quaternion analysis; and we append for comparison easy solutions obtained by methods specially adapted to each case. The advanced student need therefore pay but little attention to the next ten sections.

162. *Example I.*

Let $\phi\rho = V.\alpha\rho\beta = \gamma.$

Then $$\phi'\rho = V.\beta\rho\alpha = \phi\rho.$$

Hence $$m = \frac{1}{S.\lambda\mu\nu}\,S\,(V.\alpha\lambda\beta\;V.\alpha\mu\beta\;V.\alpha\nu\beta).$$

Now $\lambda$, $\mu$, $\nu$ are any three non-coplanar vectors; and we may therefore put for them $\alpha$, $\beta$, $\gamma$, *if the latter be non-coplanar.*

With this proviso

$$m = \frac{1}{S.\alpha\beta\gamma}\,S\,(\alpha^2\beta\,.\,\alpha\beta^2\,.\,V.\alpha\gamma\beta)$$
$$= \alpha^2\beta^2 S\alpha\beta,$$

$$m_1 = \frac{1}{S.\alpha\beta\gamma}\,S\,(\alpha\,.\,\alpha\beta^2\,.\,V.\alpha\gamma\beta + \alpha^2\beta\,.\,\beta\,.\,V.\alpha\gamma\beta + \alpha^2\beta\,.\,\alpha\beta^2\,.\,\gamma)$$
$$= -\alpha^2\beta^2,$$

$$m_2 = \frac{1}{S.\alpha\beta\gamma}\,S\,(\alpha\beta\;V.\alpha\gamma\beta + \alpha^2\beta\,.\,\beta\,.\,\gamma + \alpha\,.\,\alpha\beta^2\,.\,\gamma)$$
$$= -S\alpha\beta.$$

Hence we have by (5) above

$$\alpha^2\beta^2 S\alpha\beta\,.\,\phi^{-1}\gamma = \alpha^2\beta^2 S\alpha\beta\,.\,\rho = -\alpha^2\beta^2\gamma + S\alpha\beta\;V.\alpha\gamma\beta + V.\alpha\,(V.\alpha\gamma\beta)\,\beta,$$

which is one form of solution.

By expanding the vectors of products we may easily reduce it to the form

$$\alpha^2\beta^2 S\alpha\beta \,.\, \rho = -\alpha^2\beta^2\gamma + \alpha\beta^2 S\alpha\gamma + \beta\alpha^2 S\beta\gamma,$$

or

$$\rho = \frac{\alpha^{-1} S\alpha\gamma + \beta^{-1} S\beta\gamma - \gamma}{S\alpha\beta}.$$

163. To verify this solution, we have

$$V \,.\, \alpha\rho\beta = \frac{1}{S\alpha\beta}(\beta S\alpha\gamma + \alpha S\beta\gamma - V \,.\, \alpha\gamma\beta) = \gamma,$$

which is the given equation.

164. An easier mode of arriving at the same solution, in this simple case, is as follows:—

Operating by $S \,.\, \alpha$ and $S \,.\, \beta$ on the given equation

$$V \,.\, \alpha\rho\beta = \gamma,$$

we obtain

$$\alpha^2 S\beta\rho = S\alpha\gamma,$$

$$\beta^2 S\alpha\rho = S\beta\gamma;$$

and therefore

$$\alpha S\beta\rho = \alpha^{-1} S\alpha\gamma,$$

$$\beta S\alpha\rho = \beta^{-1} S\beta\gamma.$$

But the given equation may be written

$$\alpha S\beta\rho - \rho S\alpha\beta + \beta S\alpha\rho = \gamma.$$

Substituting and transposing we get

$$\rho S\alpha\beta = \alpha^{-1} S\alpha\gamma + \beta^{-1} S\beta\gamma - \gamma,$$

which agrees with the result of § 162.

[Note that, at first sight, one might think that the value of $\rho$ should have a term with an arbitrary scalar factor added. But the *notation* $\rho$ is limited to a vector. Had the equation been written

$$V \,.\, \alpha q\beta = \gamma$$

we should have had

$$\alpha q\beta = x + \gamma, \text{ or } q = x\alpha^{-1}\beta^{-1} + \alpha^{-1}\gamma\beta^{-1}.$$

But because $q$ is to be a vector

$$Sq = 0, \text{ or } xS\alpha\beta + S \,.\, \alpha\gamma\beta = 0;$$

and, with this value of $x$, the expression for $q$ takes the form given above for $\rho$.]

165. If $\alpha$, $\beta$, $\gamma$ be coplanar, the above mode of solution is applicable, but the result may be deduced much more simply.

For (§ 101) $S \,.\, \alpha\beta\gamma = 0$, and the equation then gives $S \,.\, \alpha\beta\rho = 0$, so that $\rho$ also is coplanar with $\alpha$, $\beta$, $\gamma$.

Hence the equation may be written

$$\alpha\rho\beta=\gamma,$$

and at once

$$\rho=\alpha^{-1}\gamma\beta^{-1};$$

and this, being a vector, may be written

$$=\alpha^{-1}S\beta^{-1}\gamma+\beta^{-1}S\alpha^{-1}\gamma-\gamma S\alpha^{-1}\beta^{-1}.$$

This formula is *equivalent* to that just given, but not equal to it term by term. [The student will find it a good exercise to prove *directly* that, if $\alpha$, $\beta$, $\gamma$ are coplanar, we have

$$\frac{1}{S\alpha\beta}(\alpha^{-1}S\alpha\gamma+\beta^{-1}S\beta\gamma-\gamma)=\alpha^{-1}S\beta^{-1}\gamma+\beta^{-1}S\alpha^{-1}\gamma-\gamma S\alpha^{-1}\beta^{-1}.]$$

The conclusion that

$$S.\alpha\beta\rho=0,$$

in this case, is not necessarily true if

$$S\alpha\beta=0.$$

But then the original equation becomes

$$\alpha S\beta\rho+\beta S\alpha\rho=\gamma,$$

which is consistent with

$$S.\alpha\beta\gamma=0.$$

This equation gives

$$\gamma(\alpha^2\beta^2-S^2\alpha\beta)=\alpha\begin{vmatrix}S\alpha\gamma & S\alpha\beta\\ S\beta\gamma & \beta^2\end{vmatrix}+\beta\begin{vmatrix}S\beta\gamma & S\alpha\beta\\ S\alpha\gamma & \alpha^2\end{vmatrix},$$

by comparison of which with the given equation we find

$$S\alpha\rho \text{ and } S\beta\rho.$$

The value of $\rho$ remains therefore with an indeterminate vector part, parallel to $\alpha\beta$; i.e. it involves one arbitrary scalar.

166. *Example II.*

Let $\phi\rho=V.\alpha\beta\rho=\gamma$.

Suppose $\alpha$, $\beta$, $\gamma$ not to be coplanar, and employ them as $\lambda$, $\mu$, $\nu$ to calculate the coefficients in the equation for $\phi^{-1}$. We have

$$S.\sigma\phi\rho=S.\sigma\alpha\beta\rho=S.\rho V.\sigma\alpha\beta=S.\rho\phi'\sigma.$$

Hence $$\phi'\rho=V.\rho\alpha\beta=V.\beta\alpha\rho.$$

We have now

$$m=\frac{1}{S.\alpha\beta\gamma}S(\beta\alpha^2.\beta\alpha\beta.V.\beta\alpha\gamma)=\frac{\alpha^2\beta^2}{S.\alpha\beta\gamma}S.\alpha\beta V.\beta\alpha\gamma$$
$$=\alpha^2\beta^2 S\alpha\beta,$$

$$m_1 = \frac{1}{S\,.\,\alpha\beta\gamma} S(\alpha\,.\,\beta\alpha\beta\,.\,V\,.\,\beta\alpha\gamma + \beta\alpha^2\,.\,\beta\,.\,V\,.\,\beta\alpha\gamma + \beta\alpha^2\,.\,\beta\alpha\beta\,.\,\gamma)$$

$$= 2(S\alpha\beta)^2 + \alpha^2\beta^2,$$

$$m_2 = \frac{1}{S\,.\,\alpha\beta\gamma} S(\alpha\,.\,\beta\,.\,V\,.\,\beta\alpha\gamma + \alpha\,.\,\beta\alpha\beta\,.\,\gamma + \beta\alpha^2\,.\,\beta\,.\,\gamma)$$

$$= 3S\alpha\beta.$$

Hence by (5) of § 160

$$\alpha^2\beta^2S\alpha\beta\,.\,\phi^{-1}\gamma = \alpha^2\beta^2S\alpha\beta\,.\,\rho$$

$$= (2(S\alpha\beta)^2 + \alpha^2\beta^2)\gamma - 3S\alpha\beta V\,.\,\alpha\beta\gamma + V\,.\,\alpha\beta V\,.\,\alpha\beta\gamma,$$

which, by expanding the vectors of products, takes easily the simpler form

$$\alpha^2\beta^2S\alpha\beta\,.\,\rho = \alpha^2\beta^2\gamma - \alpha\beta^2S\alpha\gamma + 2\beta S\alpha\beta S\alpha\gamma - \beta\alpha^2S\beta\gamma.$$

167. To verify this, operate by $V\,.\,\alpha\beta$ on both sides, and we have

$$\alpha^2\beta^2S\alpha\beta V.\alpha\beta\rho = \alpha^2\beta^2V.\,\alpha\beta\gamma - V.\,\alpha\beta\alpha\beta^2S\alpha\gamma + 2\alpha\beta^2S\alpha\beta S\alpha\gamma - \alpha\alpha^2\beta^2S\beta\gamma$$

$$= \alpha^2\beta^2(\alpha S\beta\gamma - \beta S\alpha\gamma + \gamma S\alpha\beta) - (2\alpha S\alpha\beta - \beta\alpha^2)\beta^2S\alpha\gamma$$
$$+ 2\alpha\beta^2S\alpha\beta S\alpha\gamma - \alpha\alpha^2\beta^2S\beta\gamma$$

$$= \alpha^2\beta^2S\alpha\beta\,.\,\gamma,$$

or
$$V\,.\,\alpha\beta\rho = \gamma.$$

168. To solve the same equation without employing the general method, we may proceed as follows:—

$$\gamma = V\,.\,\alpha\beta\rho = \rho S\alpha\beta + V\,.\,V(\alpha\beta)\,\rho.$$

Operating by $S\,.\,V\alpha\beta$ we have

$$S\,.\,\alpha\beta\gamma = S\,.\,\alpha\beta\rho S\alpha\beta.$$

Divide this by $S\alpha\beta$, and add it to the given equation. We thus obtain

$$\gamma + \frac{S\,.\,\alpha\beta\gamma}{S\alpha\beta} = \rho S\alpha\beta + V.V(\alpha\beta)\,\rho + S\,.\,V(\alpha\beta)\,\rho,$$

$$= (S\alpha\beta + V\alpha\beta)\,\rho,$$

$$= \alpha\beta\rho.$$

Hence
$$\rho = \beta^{-1}\alpha^{-1}\left(\gamma + \frac{S\,.\,\alpha\beta\gamma}{S\alpha\beta}\right),$$

a form of solution somewhat simpler than that before obtained.

To shew that they agree, however, let us multiply by $\alpha^2\beta^2S\alpha\beta$, and we get

$$\alpha^2\beta^2S\alpha\beta\,.\,\rho = \beta\alpha\gamma S\alpha\beta + \beta\alpha S\,.\,\alpha\beta\gamma.$$

In this form we see at once that the right-hand side is a vector, since its scalar is evidently zero (§ 89). Hence we may write

$$\alpha^2\beta^2 S\alpha\beta . \rho = V. \beta\alpha\gamma S\alpha\beta - V\alpha\beta S . \alpha\beta\gamma.$$

But by (3) of § 91,

$$-\gamma S . \alpha\beta V\alpha\beta + \alpha S . \beta (V\alpha\beta)\gamma + \beta S . V(\alpha\beta)\alpha\gamma + V\alpha\beta S . \alpha\beta\gamma = 0.$$

Add this to the right-hand side, and we have

$$\alpha^2\beta^2 S\alpha\beta . \rho = \gamma ((S\alpha\beta)^2 - S . \alpha\beta V\alpha\beta) - \alpha (S\alpha\beta S\beta\gamma - S . \beta (V\alpha\beta)\gamma)$$
$$+ \beta (S\alpha\beta S\alpha\gamma + S . V(\alpha\beta)\alpha\gamma).$$

But $$(S\alpha\beta)^2 - S . \alpha\beta V\alpha\beta = (S\alpha\beta)^2 - (V\alpha\beta)^2 = \alpha^2\beta^2,$$

$$S\alpha\beta S\beta\gamma - S . \beta (V\alpha\beta)\gamma = S\alpha\beta S\beta\gamma - S\beta\alpha S\beta\gamma + \beta^2 S\alpha\gamma = \beta^2 S\alpha\gamma,$$

$$S\alpha\beta S\alpha\gamma + S . V(\alpha\beta)\alpha\gamma = S\alpha\beta S\alpha\gamma + S\alpha\beta S\alpha\gamma - \alpha^2 S\beta\gamma$$
$$= 2S\alpha\beta S\alpha\gamma - \alpha^2 S\beta\gamma ;$$

and the substitution of these values renders our equation identical with that of § 166.

[If $\alpha$, $\beta$, $\gamma$ be coplanar, the simplified forms of the expression for $\rho$ lead to the equation

$$S\alpha\beta . \beta^{-1}\alpha^{-1}\gamma = \gamma - \alpha^{-1} S\alpha\gamma + 2\beta S\alpha^{-1}\beta^{-1} S\alpha\gamma - \beta^{-1} S\beta\gamma,$$

which, as before, we leave as an exercise to the student.]

169. *Example III.* The solution of the equation

$$V\epsilon\rho = \gamma$$

leads to the vanishing of some of the quantities $m$. Before, however, treating it by the general method, we shall deduce its solution from that of

$$V. \alpha\beta\rho = \gamma$$

already given. Our reason for so doing is that we thus have an opportunity of shewing the nature of some of the cases in which one or more of $m$, $m_1$, $m_2$ vanish; and also of introducing an example of the use of vanishing fractions in quaternions. Far simpler solutions will be given in the following sections.

The solution of the last-written equation is, § 166,

$$\alpha^2\beta^2 S\alpha\beta . \rho = \alpha^2\beta^2\gamma - \alpha\beta^2 S\alpha\gamma - \beta\alpha^2 S\beta\gamma + 2\beta S\alpha\beta S\alpha\gamma.$$

If we now put $$\alpha\beta = e + \epsilon$$

where $e$ is a scalar, the solution of the first-written equation will evidently be derived from that of the second by making $e$ gradually tend to zero.

We have, for this purpose, the following necessary transformations:—

$$\alpha^2\beta^2 = \alpha\beta\, K.\alpha\beta = (e+\epsilon)(e-\epsilon) = e^2 - \epsilon^2,$$

$$\begin{aligned}\alpha\beta^2 S\alpha\gamma + \beta\alpha^2 S\beta\gamma &= \alpha\beta\,.\,\beta S\alpha\gamma + \beta\alpha\,.\,\alpha S\beta\gamma,\\ &= (e+\epsilon)\,\beta S\alpha\gamma + (e-\epsilon)\,\alpha S\beta\gamma,\\ &= e\,(\beta S\alpha\gamma + \alpha S\beta\gamma) + \epsilon V.\gamma V\alpha\beta,\\ &= e\,(\beta S\alpha\gamma + \alpha S\beta\gamma) + \epsilon V\gamma\epsilon.\end{aligned}$$

Hence the solution becomes

$$\begin{aligned}(e^2-\epsilon^2)\,e\rho &= (e^2-\epsilon^2)\,\gamma - e\,(\beta S\alpha\gamma + \alpha S\beta\gamma) - \epsilon V\gamma\epsilon + 2e\beta S\alpha\gamma,\\ &= (e^2-\epsilon^2)\,\gamma + eV.\gamma V\alpha\beta - \epsilon V\gamma\epsilon,\\ &= (e^2-\epsilon^2)\,\gamma + eV\gamma\epsilon + \gamma\epsilon^2 - \epsilon S\gamma\epsilon,\\ &= e^2\gamma + eV\gamma\epsilon - \epsilon S\gamma\epsilon.\end{aligned}$$

Dividing by $e$, and then putting $e=0$, we have

$$-\epsilon^2\rho = V\gamma\epsilon - \epsilon\mathcal{L}_0\left(\frac{S\gamma\epsilon}{e}\right).$$

Now, by the form of the given equation, we see that

$$S\gamma\epsilon = 0.$$

Hence the limit is indeterminate, and we may put for it $x$, where $x$ is *any* scalar. Our solution is, therefore,

$$\rho = -V\frac{\gamma}{\epsilon} + x\epsilon^{-1};$$

or, as it may be written, since $S\gamma\epsilon = 0$,

$$\rho = \epsilon^{-1}(\gamma + x).$$

The verification is obvious—for we have

$$\epsilon\rho = \gamma + x.$$

170. This suggests a very simple mode of solution. For we see that the given equation leaves $S\epsilon\rho$ indeterminate. Assume, therefore,

$$S\epsilon\rho = x$$

and add to the given equation. We obtain

$$\epsilon\rho = x + \gamma,$$

or

$$\rho = \epsilon^{-1}(\gamma + x),$$

if, and only if, $\rho$ satisfies the equation

$$V\epsilon\rho = \gamma.$$

171. To apply the general method, we may take $\epsilon$, $\gamma$ and $\epsilon\gamma$ (which is a vector) for $\lambda$, $\mu$, $\nu$.

We find $\phi'\rho = V\rho\epsilon.$

Hence

$$m = 0,$$

$$m_1 = -\frac{1}{\epsilon^2\gamma^2} S.(\epsilon.\epsilon\gamma.\epsilon^2\gamma) = -\epsilon^2,$$

$$m_2 = 0.$$

Hence

$$-\epsilon^2\phi + \phi^3 = 0,$$

or

$$\phi^{-1} = \frac{1}{\epsilon^2}\phi + \phi^{-2}0.$$

That is,

$$\rho = \frac{1}{\epsilon^2} V\epsilon\gamma + x\epsilon,$$

$$= \epsilon^{-1}\gamma + x\epsilon, \text{ as before.}$$

Our warrant for putting $x\epsilon$, as the equivalent of $\phi^{-2}0$ is this:—

The equation $\phi^2\sigma = 0$

may be written $V.\epsilon V\epsilon\sigma = 0 = \sigma\epsilon^2 - \epsilon S\epsilon\sigma.$

Hence, unless $\sigma = 0$, we have $\sigma \parallel \epsilon = x\epsilon.$

[Of course it is well to avoid, when possible, the use of expressions such as $\phi^{-2}0$ &c. but the student must be prepared to meet them; and it is well that he should gain confidence in using them, by verifying that they lead to correct results in cases where other modes of solution are available.]

172. *Example IV.* As a final example let us take the most general form of $\phi$, which, as has been shewn in § 151, may be expressed as follows:—

$$\phi\rho = \alpha S\beta\rho + \alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho = \gamma.$$

Here $\phi'\rho = \beta S\alpha\rho + \beta_1 S\alpha_1\rho + \beta_2 S\alpha_2\rho,$

and, consequently, taking $\alpha$, $\alpha_1$, $\alpha_2$, which are in this case non-coplanar vectors, for $\lambda$, $\mu$, $\nu$, we have

$$m = \frac{1}{S.\alpha\alpha_1\alpha_2} S.(\beta S\alpha\alpha + \beta_1 S\alpha_1\alpha + \beta_2 S\alpha_2\alpha)(\beta S\alpha\alpha_1 + \beta_1 S\alpha_1\alpha_1 + \ldots)$$
$$(\beta S\alpha\alpha_2 + \ldots\ldots)$$

$$= \frac{S.\beta\beta_1\beta_2}{S.\alpha\alpha_1\alpha_2}\begin{vmatrix} S\alpha\alpha & S\alpha_1\alpha & S\alpha_2\alpha \\ S\alpha\alpha_1 & S\alpha_1\alpha_1 & S\alpha_2\alpha_1 \\ S\alpha\alpha_2 & S\alpha_1\alpha_2 & S\alpha_2\alpha_2 \end{vmatrix}$$

$$= \frac{S.\beta\beta_1\beta_2}{S.\alpha\alpha_1\alpha_2}{}^2 (AS\alpha\alpha + A_1 S\alpha_1\alpha + A_2 S\alpha_2\alpha),$$

where

$$A = S\alpha_1\alpha_1 S\alpha_2\alpha_2 - S\alpha_2\alpha_1 S\alpha_1\alpha_2$$

$$= -S.V\alpha_1\alpha_2 V\alpha_1\alpha_2$$

$$\begin{aligned} A_1 &= S\alpha_2\alpha_1 S\alpha\alpha_2 - S\alpha\alpha_1 S\alpha_2\alpha_2 \\ &= -S\,.\,V\alpha_2\alpha V\alpha_1\alpha_2 \\ A_2 &= S\alpha\alpha_1 S\alpha_1\alpha_2 - S\alpha_1\alpha_1 S\alpha\alpha_2 \\ &= -S\,.\,V\alpha\alpha_1 V\alpha_1\alpha_2. \end{aligned}$$

Hence the value of the determinant is

$$\begin{aligned} &-(S\alpha\alpha S\,.\,V\alpha_1\alpha_2 V\alpha_1\alpha_2 + S\alpha_1\alpha S\,.\,V\alpha_2\alpha V\alpha_1\alpha_2 + S\alpha_2\alpha S\,.\,V\alpha\alpha_1 V\alpha_1\alpha_2) \\ &\qquad = -S\,.\,\alpha\,(V\alpha_1\alpha_2 S\,.\,\alpha\alpha_1\alpha_2)\,\{\text{by § 92 (4)}\} = -(S\,.\,\alpha\alpha_1\alpha_2)^2. \end{aligned}$$

The interpretation of this result in spherical trigonometry is very interesting. (See Ex. (9) p. 90.)

By it we see that

$$m = -S\,.\,\alpha\alpha_1\alpha_2 S\,.\,\beta\beta_1\beta_2.$$

Similarly,

$$\begin{aligned} m_1 &= \frac{1}{S\,.\,\alpha\alpha_1\alpha_2} S\,.\,[\alpha\,(\beta S\alpha\alpha_1 + \beta_1 S\alpha_1\alpha_1 + \beta_2 S\alpha_2\alpha_1) \\ &\qquad\qquad (\beta S\alpha\alpha_2 + \beta_1 S\alpha_1\alpha_2 + \beta_2 S\alpha_2\alpha_2) + \&\text{c.}] \\ &= \frac{1}{S\,.\,\alpha\alpha_1\alpha_2} (S\,.\,\alpha\beta\beta_1 (S\alpha\alpha_1 S\alpha_1\alpha_2 - S\alpha_1\alpha_1 S\alpha\alpha_2) + \ldots) \\ &= \frac{1}{S\,.\,\alpha\alpha_1\alpha_2} (S\,.\,\alpha\beta\beta_1 S\,.\,\alpha V\,.\,\alpha_1 V\alpha_2\alpha_1 + \ldots) \\ &= -\frac{1}{S\,.\,\alpha\alpha_1\alpha_2} [S\,.\,\alpha\,(V\beta\beta_1 S\,.\,V\alpha\alpha_1 V\alpha_1\alpha_2 + V\beta_2\beta S\,.\,V\alpha_2\alpha V\alpha_1\alpha_2 \\ &\qquad\qquad + V\beta_1\beta_2 S\,.\,V\alpha_1\alpha_2 V\alpha_1\alpha_2) \\ &\qquad + S\,.\,\alpha_1 (V\beta\beta_1 S\,.\,V\alpha\alpha_1 V\alpha_2\alpha + \ldots) \\ &\qquad + S\,.\,\alpha_2 (V\beta\beta_1 S\,.\,V\alpha\alpha_1 V\alpha\alpha_1 + \ldots)]\,; \end{aligned}$$

or, taking the terms by columns instead of by rows,

$$\begin{aligned} &= -\frac{1}{S\,.\,\alpha\alpha_1\alpha_2} [S.V\beta\beta_1 (\alpha S.V\alpha\alpha_1 V\alpha_1\alpha_2 + \alpha_1 S.V\alpha\alpha_1 V\alpha_2\alpha + \alpha_2 S\,.\,V\alpha\alpha_1 V\alpha\alpha_1) \\ &\qquad\qquad + \;.\;\;.\;\;.\;\;.\;\;.\; + \;.\;\;.\;\;.\;\;.\; ], \\ &= -\frac{1}{S\,.\,\alpha\alpha_1\alpha_2} [S\,.\,V\beta\beta_1 (V\alpha\alpha_1 S\,.\,\alpha\alpha_1\alpha_2) \;+\; .\;\;.\;\;.\;\;. \;], \\ &= -S\,(V\alpha\alpha_1 V\beta\beta_1 + V\alpha_1\alpha_2 V\beta_1\beta_2 + V\alpha_2\alpha V\beta_2\beta). \end{aligned}$$

Again,

$$\begin{aligned} m_2 &= \frac{1}{S\,.\,\alpha\alpha_1\alpha_2} S\,[\alpha\alpha_1 (\beta S\alpha\alpha_2 + \beta_1 S\alpha_1\alpha_2 + \ldots) + \alpha_2\alpha\,(\beta S\alpha\alpha_1 + \ldots) \\ &\qquad\qquad + \alpha_1\alpha_2\,(\beta S\alpha\alpha + \ldots)], \end{aligned}$$

or, grouping as before,

$$= \frac{1}{S\,.\,\alpha\alpha_1\alpha_2} S\,[\beta\,(V\alpha\alpha_1 S\alpha\alpha_2 + V\alpha_2\alpha S\alpha\alpha_1 + V\alpha_1\alpha_2 S\alpha\alpha) + \ldots],$$

$$= \frac{1}{S\,.\,\alpha\alpha_1\alpha_2} S[\beta(\alpha S\,.\,\alpha\alpha_1\alpha_2) + \ldots] \text{ (§ 92 (4))},$$
$$= S(\gamma\beta + \alpha_1\beta_1 + \alpha_2\beta_2).$$

And the solution is, therefore,

$$\phi^{-1}\gamma S\,.\,\alpha\alpha_1\alpha_2 S\,.\,\beta\beta_1\beta_2 = \rho S\,.\,\alpha\alpha_1\alpha_2 S\,.\,\beta\beta_1\beta_2$$
$$= \gamma\Sigma S\,.\,V\alpha\alpha_1 V\beta\beta_1 + \phi\gamma\Sigma S\alpha\beta - \phi^2\gamma.$$

[It will be excellent practice for the student to work out in detail the blank portions of the above investigation, and also to prove directly that the value of $\rho$ we have just found satisfies the given equation.]

173. But it is not necessary to go through such a long process to get the solution—though it will be advantageous to the student to read it carefully—for if we operate on the proposed equation by $S\,.\,\alpha_1\alpha_2$, $S\,.\,\alpha_2\alpha$, and $S\,.\,\alpha\alpha_1$ we get

$$S\,.\,\alpha_1\alpha_2\alpha S\beta\rho = S\,.\,\alpha_1\alpha_2\gamma,$$
$$S\,.\,\alpha_2\alpha\alpha_1 S\beta_1\rho = S\,.\,\alpha_2\alpha\gamma,$$
$$S\,.\,\alpha\alpha_1\alpha_2 S\beta_2\rho = S\,.\,\alpha\alpha_1\gamma.$$

From these, by § 92 (4), we have at once

$$\rho S\,.\,\alpha\alpha_1\alpha_2 S\,.\,\beta\beta_1\beta_2 = V\beta\beta_1 S\,.\,\alpha\alpha_1\gamma + V\beta_1\beta_2 S\,.\,\alpha_1\alpha_2\gamma + V\beta_2\beta S\,.\,\alpha_2\alpha\gamma.$$

The student will find it a useful exercise to prove that this is equivalent to the solution in § 172.

To verify the present solution we have

$$(\alpha S\beta\rho + \alpha_1 S\beta_1\rho + \alpha_2 S\beta_2\rho)\, S\,.\,\alpha\alpha_1\alpha_2 S\,.\,\beta\beta_1\beta_2$$
$$= \alpha S\,.\,\beta\beta_1\beta_2 S\,.\,\alpha_1\alpha_2\gamma + \alpha_1 S\,.\,\beta_1\beta_2\beta S\,.\,\alpha_2\alpha\gamma + \alpha_2 S\,.\,\beta_2\beta\beta_1 S\,.\,\alpha\alpha_1\gamma$$
$$= S\,.\,\beta\beta_1\beta_2(\gamma S\,.\,\alpha\alpha_1\alpha_2), \text{ by § 91 (3)}.$$

174. It is evident, from these examples, that for special cases we can usually find modes of solution of the linear and vector equation which are simpler in application than the general process of § 160. The real value of that process however consists partly in its enabling us to express inverse functions of $\phi$, such as $(\phi - g)^{-1}$ for instance, in terms of direct operations, a property which will be of great use to us later; partly in its leading us to the fundamental cubic

$$\phi^3 - m_2\phi^2 + m_1\phi - m = 0,$$

which is an immediate deduction from the equation of § 160, and whose interpretation is of the utmost importance with reference to the axes of surfaces of the second order, principal axes of inertia, the analysis of strains in a distorted solid, and various similar enquiries.

We see, of course, that the existence of the cubic renders it a mere question of ordinary algebra to express any rational function whatever of $\phi$ in a rational three-term form such as

$$A+B\phi+C\phi^2.$$

In fact it will be seen to follow, from the results of § 177 below, that we have in general three independent scalar equations of the form

$$F(g)=A+Bg+Cg^2,$$

which determine the values of $A$, $B$, $C$ without ambiguity. The result appears in a form closely resembling that known as Lagrange's interpolation formula.

175. When the function $\phi$ *is its own conjugate*, that is, when

$$S\rho\phi\sigma=S\sigma\phi\rho$$

for all values of $\rho$ and $\sigma$, the vectors for which

$$(\phi-g)\,\rho=0$$

form in general a real and definite rectangular system. This, of course, may in particular cases degrade into one definite vector, and *any* pair of others perpendicular to it; and cases may occur in which the equation is satisfied for *every* vector.

To prove this, suppose the roots of

$$m_g=m-m_1g+m_2g^2-g^3=0$$

(§ 159 (4)) to be real and different, then

$$\left.\begin{aligned}\phi\rho_1&=g_1\rho_1\\ \phi\rho_2&=g_2\rho_2\\ \phi\rho_3&=g_3\rho_3\end{aligned}\right\}$$

where $\rho_1$, $\rho_2$, $\rho_3$ are three definite vectors determined by the constants involved in $\phi$.

Hence, operating on the first by $S\rho_2$, and on the second by $S\rho_1$, we have

$$S\,.\,\rho_2\phi\rho_1=g_1S\rho_1\rho_2,$$
$$S\,.\,\rho_1\phi\rho_2=g_2S\rho_1\rho_2.$$

The first members of these equations are equal, because $\phi$ is its own conjugate.

Thus $$(g_1-g_2)\,S\rho_1\rho_2=0;$$

which, as $g_1$ and $g_2$ are by hypothesis different, requires

$$S\rho_1\rho_2=0.$$

Similarly $S\rho_2\rho_3=0, \qquad S\rho_3\rho_1=0.$

If two roots be equal, as $g_2, g_3$, we still have, by the above proof, $S\rho_1\rho_2=0$ and $S\rho_1\rho_3=0$. But there is nothing farther to determine $\rho_2$ and $\rho_3$, which are therefore *any* vectors perpendicular to $\rho_1$.

If all three roots be equal, *every* real vector satisfies the equation $(\phi-g)\rho=0$.

176. Next as to the *reality* of the three directions in this case.

Suppose $g+h\sqrt{-1}$ to be a root, and let $\rho+\sigma\sqrt{-1}$ be the corresponding value of $\rho$, where $g$ and $h$ are real numbers, $\rho$ and $\sigma$ real vectors, and $\sqrt{-1}$ the old imaginary of algebra.

Then $\phi(\rho+\sigma\sqrt{-1})=(g+h\sqrt{-1})(\rho+\sigma\sqrt{-1}),$

and this divides itself, as in algebra, into the two equations

$$\phi\rho=g\rho-h\sigma,$$
$$\phi\sigma=h\rho+g\sigma.$$

Operating on these by $S.\sigma$, $S.\rho$ respectively, and taking the difference of the results, remembering our condition as to the nature of $\phi$

$$S\sigma\phi\rho=S\rho\phi\sigma,$$

we have $$h(\sigma^2+\rho^2)=0.$$

But, as $\sigma$ and $\rho$ are both real vectors, the sum of their squares cannot vanish, unless their tensors separately vanish. Hence $h$ vanishes, and with it the impossible part of the root.

The function $\phi$ need not be self-conjugate, in order that the roots of

$$m_g=0$$

may be all real. For we may take $g_1, g_2, g_3$ any real scalars, and $\alpha, \beta, \gamma$ any three real, non-coplanar, vectors. Then if $\phi$ be such that

$$S.\alpha\beta\gamma.\phi\rho=g_1\alpha S.\beta\gamma\rho+g_2\beta S.\gamma\alpha\rho+g_3\gamma S.\alpha\beta\rho,$$

we have obviously

$$(\phi-g_1)\alpha=0,\ (\phi-g_2)\beta=0,\ (\phi-g_3)\gamma=0.$$

Here $\phi$ is self-conjugate *only* if $\alpha, \beta, \gamma$ form a rectangular system.

177. Thus though we have shewn that the equation

$$g^3-m_2g^2+m_1g-m=0$$

has three real roots, in general different from one another, when $\phi$

is self-conjugate, the converse is by no means true. This must be most carefully kept in mind.

In all cases the cubic in $\phi$ may be written

$$(\phi - g_1)(\phi - g_2)(\phi - g_3) = 0 \dots\dots\dots\dots\dots\dots (1),$$

and in this form we can easily see its meaning, provided the values of $g$ are real. For there are in every such case three real (and in general non-coplanar) vectors, $\rho_1$, $\rho_2$, $\rho_3$ for which respectively

$$(\phi - g_1)\rho_1 = 0, \quad (\phi - g_2)\rho_2 = 0, \quad (\phi - g_3)\rho_3 = 0.$$

Then, since any vector $\rho$ may be expressed by the equation

$$\rho S . \rho_1\rho_2\rho_3 = \rho_1 S . \rho_2\rho_3\rho + \rho_2 S . \rho_3\rho_1\rho + \rho_3 S . \rho_1\rho_2\rho \text{ (§ 91)},$$

we see that when the complex operation, denoted by the left-hand member of the symbolic equation, (1), is performed on $\rho$, the first of the three factors makes the term in $\rho_1$ vanish, the second and third those in $\rho_2$ and $\rho_3$ respectively. In other words, by the successive performance, upon a vector, of the operations $\phi - g_1$, $\phi - g_2$, $\phi - g_3$, it is deprived successively of its resolved parts in the directions of $\rho_1$, $\rho_2$, $\rho_3$ respectively; and is thus necessarily reduced to zero, since $\rho_1$, $\rho_2$, $\rho_3$ are (because we have supposed $g_1$, $g_2$, $g_3$ to be distinct) distinct and non-coplanar vectors.

178. If we take $\rho_1$, $\rho_2$, $\rho_3$ as rectangular *unit*-vectors, we have

$$-\rho = \rho_1 S\rho_1\rho + \rho_2 S\rho_2\rho + \rho_3 S\rho_3\rho,$$

whence $\phi\rho = -g_1\rho_1 S\rho_1\rho - g_2\rho_2 S\rho_2\rho - g_3\rho_3 S\rho_3\rho$;

or, still more simply, putting $i, j, k$ for $\rho_1$, $\rho_2$, $\rho_3$, we find that *any* self-conjugate function may be thus expressed

$$\phi\rho = -g_1 iSi\rho - g_2 jSj\rho - g_3 kSk\rho \dots\dots\dots\dots (2),$$

provided, of course, $i, j, k$ be taken as the roots of the equation

$$V\rho\phi\rho = 0.$$

A rectangular unit-vector system requires three scalar quantities, only, for its full specification. $g_1$, $g_2$, $g_3$ are other three. Thus any self-conjugate function involves only *six* independent scalars.

179. A very important transformation of the self-conjugate linear and vector function is easily derived from this form.

We have seen that it involves, besides those of the system $i, j, k$, *three* scalar constants only, viz. $g_1$, $g_2$, $g_3$. Let us enquire, then, whether it can be reduced to the following form

$$\phi\rho = f\rho + hV . (i + ek)\rho(i - ek) \dots\dots\dots\dots (3),$$

which also involves but three scalar constants $f$, $h$, $e$, in addition to those of $i$, $j$, $k$, the roots of

$$V\rho\phi\rho = 0.$$

Substituting for $\rho$ the equivalent

$$\rho = -iSi\rho - jSj\rho - kSk\rho,$$

expanding, and equating coefficients of $i, j, k$ in the two expressions (2) and (3) for $\phi\rho$, we find

$$g_1 = f - h(1+e^2),$$
$$g_2 = f + h(1-e^2),$$
$$g_3 = f + h(1+e^2).$$

These give at once

$$-(g_1 - g_2) = 2h,$$
$$-(g_2 - g_3) = 2e^2h.$$

Hence, as we suppose the transformation to be real, and therefore $e^2$ to be positive, it is evident that $g_1 - g_2$ and $g_2 - g_3$ have the same sign; so that we must choose as auxiliary vectors in the last term of $\phi\rho$ those two of the rectangular directions $i, j, k$ for which the coefficients $g$ have respectively the greatest and least values.

We have then

$$e^2 = \frac{g_2 - g_3}{g_1 - g_2},$$
$$h = -\tfrac{1}{2}(g_1 - g_2),$$

and

$$f = \tfrac{1}{2}(g_1 + g_3).$$

180. We may, therefore, always determine definitely the vectors $\lambda$, $\mu$, and the scalar $f$, in the equation

$$\phi\rho = f\rho + V.\lambda\rho\mu$$

when $\phi$ is self-conjugate, and the corresponding cubic has not equal roots; subject to the single restriction that

$$T.\lambda\mu$$

is known, but not the separate tensors of $\lambda$ and $\mu$. This result is important in the theory of surfaces of the second order, and in that of Fresnel's Wave-Surface, and will be considered in Chapters IX. and XII.

181. Another important transformation of $\phi$ when self-conjugate is the following,

$$\phi\rho = a\alpha V\alpha\rho + b\beta S\beta\rho,$$

where $a$ and $b$ are scalars, and $\alpha$ and $\beta$ unit-vectors. This, of course, involves six scalar constants, and belongs to the most general form

$$\phi\rho = -g_1\rho_1 S\rho_1\rho - g_2\rho_2 S\rho_2\rho - g_3\rho_3 S\rho_3\rho,$$

where $\rho_1, \rho_2, \rho_3$ are the rectangular unit-vectors for which $\rho$ and $\phi\rho$ are parallel. We merely mention this form in passing, as it belongs to the *focal* transformation of the equation of surfaces of the second order, which will not be farther alluded to in this work. It will be a good exercise for the student to determine $\alpha$, $\beta$, $a$ and $b$, in terms of $g_1, g_2, g_3$, and $\rho_1, \rho_2, \rho_3$.

182. We cannot afford space for a detailed account of the singular properties of these vector functions, and will therefore content ourselves with the enuntiation and proof of one or two of the more important.

In the equation $m\phi^{-1}V\lambda\mu = V\phi'\lambda\phi'\mu$ (§ 157), substitute $\lambda$ for $\phi'\lambda$ and $\mu$ for $\phi'\mu$, and we have

$$mV\phi'^{-1}\lambda\phi'^{-1}\mu = \phi V\lambda\mu.$$

Change $\phi$ to $\phi - g$, and therefore $\phi'$ to $\phi' - g$, and $m$ to $m_g$, we have

$$m_g V(\phi' - g)^{-1}\lambda(\phi' - g)^{-1}\mu = (\phi - g)\, V\lambda\mu;$$

a formula which will be found to be of considerable use.

183. Again, by § 159,

$$\frac{m_g}{g} S.\rho(\phi - g)^{-1}\rho = \frac{m}{g} S\rho\phi^{-1}\rho - S\rho\chi\rho + g\rho^2.$$

Similarly $\dfrac{m_h}{h} S.\rho(\phi - h)^{-1}\rho = \dfrac{m}{h} S\rho\phi^{-1}\rho - S\rho\chi\rho + h\rho^2.$

Hence

$$\frac{m_g}{g} S.\rho(\phi - g)^{-1}\rho - \frac{m_h}{h} S.\rho(\phi - h)^{-1}\rho = (g - h)\left\{\rho^2 - \frac{mS\rho\phi^{-1}\rho}{gh}\right\}.$$

That is, the functions

$$\frac{m_g}{g} S.\rho(\phi - g)^{-1}\rho, \quad \text{and} \quad \frac{m_h}{h} S.\rho(\phi - h)^{-1}\rho$$

are identical, i.e. *when equated to constants represent the same series of surfaces*, not merely when

$$g = h,$$

but also, whatever be $g$ and $h$, if they be scalar functions of $\rho$ which satisfy the equation

$$mS.\rho\phi^{-1}\rho = gh\rho^2.$$

This is a generalization, due to Hamilton, of a singular result obtained by the author*.

184. It is easy to extend these results; but, for the benefit of beginners, we may somewhat simplify them. Let us confine our attention to cones, with equations such as

$$\left.\begin{array}{l} S \,.\, \rho(\phi - g)^{-1}\rho = 0, \\ S \,.\, \rho(\phi - h)^{-1}\rho = 0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

These are equivalent to $\quad mS\rho\phi^{-1}\rho - gS\rho\chi\rho + g^2\rho^2 = 0,$

$$mS\rho\phi^{-1}\rho - hS\rho\chi\rho + h^2\rho^2 = 0.$$

Hence

$$m(1-x)\,S\rho\phi^{-1}\rho - (g - hx)\,S\rho\chi\rho + (g^2 - h^2x)\,\rho^2 = 0,$$

whatever scalar be represented by $x$.

That is, the two equations (1) represent the *same* surface if this identity be satisfied. As particular cases let

(1) $x = 1$, in which case

$$-S\rho^{-1}\chi\rho + g + h = 0.$$

(2) $g - hx = 0$, in which case

$$m\left(1 - \frac{g}{h}\right) S\rho^{-1}\phi^{-1}\rho + \left(g^2 - h^2\frac{g}{h}\right) = 0,$$

or

$$mS\rho^{-1}\phi^{-1}\rho - gh = 0.$$

(3) $x = \dfrac{g^2}{h^2}$, giving

$$-m\left(1 - \frac{g^2}{h^2}\right) S\rho\phi^{-1}\rho + \left(g - h\frac{g^2}{h^2}\right) S\rho\chi\rho = 0,$$

or

$$-m(h+g)\,S\rho\phi^{-1}\rho + ghS\rho\chi\rho = 0.$$

185. In various investigations we meet with the quaternion

$$q = \alpha\phi\alpha + \beta\phi\beta + \gamma\phi\gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\alpha$, $\beta$, $\gamma$ are three unit-vectors at right angles to each other. It admits of being put in a very simple form, which is occasionally of considerable importance.

We have, obviously, by the properties of a rectangular unit-system

$$q = \beta\gamma\phi\alpha + \gamma\alpha\phi\beta + \alpha\beta\phi\gamma.$$

As we have also

$$S \,.\, \alpha\beta\gamma = -1 \quad (\S\, 71\ (13)),$$

---

* Note on the Cartesian equation of the Wave-Surface. *Quarterly Math. Journal*, Oct. 1859.

a glance at the formulæ of § 159 shews that

$$Sq = -m_2,$$

at least if $\phi$ be self-conjugate. Even if it be not, still (as will be shewn in § 186) the term by which it differs from a self-conjugate function is of such a form that it disappears in $Sq$.

We have also, by § 90 (2),

$$\begin{aligned} Vq &= \alpha (S\beta\phi\gamma - S\gamma\phi\beta) + \beta (S\gamma\phi\alpha - S\alpha\phi\gamma) + \gamma (S\alpha\phi\beta - S\beta\phi\alpha) \\ &= \alpha S\beta (\phi - \phi')\gamma + \beta S\gamma (\phi - \phi')\alpha + \gamma S\alpha (\phi - \phi')\beta \\ &= \alpha S . \beta\epsilon\gamma + \beta S . \gamma\epsilon\alpha + \gamma S . \alpha\epsilon\beta \quad (\S\ 186) \\ &= -(\alpha S\alpha\epsilon + \beta S\beta\epsilon + \gamma S\gamma\epsilon) = \epsilon. \end{aligned}$$

[We may note in passing that the quaternion (1) admits of being expressed in the remarkable forms

$$\nabla\phi\rho, \text{ or } K . \phi(\nabla)\rho;$$

where (§ 145) $$\nabla = \alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz},$$

and $$\rho = \alpha x + \beta y + \gamma z.$$

We will recur to this towards the end of the work.]

Many similar singular properties of $\phi$ in connection with a rectangular system might easily be given; for instance,

$$\begin{aligned} &V(\alpha V\phi\beta\phi\gamma + \beta V\phi\gamma\phi\alpha + \gamma V\phi\alpha\phi\beta) \\ &\qquad = mV(\alpha\phi'^{-1}\alpha + \beta\phi'^{-1}\beta + \gamma\phi'^{-1}\gamma) = mV . \nabla\phi'^{-1}\rho = \phi\epsilon; \end{aligned}$$

which the reader may easily verify by a process similar to that just given, or (more directly) by the help of § 157 (2). A few others will be found among the Examples appended to this Chapter.

186. To conclude, we may remark that, as in many of the immediately preceding investigations we have supposed $\phi$ to be self-conjugate, a very simple step enables us to pass from this to the non-conjugate form.

For, if $\phi'$ be conjugate to $\phi$, we have

$$S\rho\phi'\sigma = S\sigma\phi\rho,$$

and also $$S\rho\phi\sigma = S\sigma\phi'\rho.$$

Adding, we have

$$S\rho (\phi + \phi')\sigma = S\sigma (\phi + \phi')\rho;$$

so that the function $(\phi + \phi')$ is self-conjugate.

Again, $$S\rho\phi\rho = S\rho\phi'\rho,$$

which gives $$S\rho (\phi - \phi')\rho = 0.$$

Hence $$(\phi - \phi')\rho = V\epsilon\rho,$$

where, if $\phi$ be not self-conjugate, $\epsilon$ is some real vector, and therefore

$$\begin{aligned}\phi\rho &= \tfrac{1}{2}(\phi+\phi')\rho + \tfrac{1}{2}(\phi-\phi')\rho\\ &= \tfrac{1}{2}(\phi+\phi')\rho + \tfrac{1}{2}V\epsilon\rho.\end{aligned}$$

Thus *every non-conjugate linear and vector function differs from a conjugate function solely by a term of the form*

$$V\epsilon\rho.$$

The geometric signification of this will be found in the Chapter on Kinematics.

The vector $\epsilon$ involves, of course, three scalar constants. Hence (§§ 151, 178) the linear and vector function involves, in general, *nine.*

187. Before leaving this part of the subject, it may be well to say a word or two as to the conditions for three real vector solutions of the equation

$$V\rho\phi\rho = 0.$$

This question is very fully treated in Hamilton's *Elements,* and also by Plarr in the *Trans. R. S. E.* For variety we adopt a semi-graphic method*, based on the result of last section. By that result we see that the equation to be solved may be written as

$$\phi\rho = \varpi\rho + V\epsilon\rho = x\rho \text{..........................}(1)$$

where $\varpi$ is a given self-conjugate function, $\epsilon$ a given vector, and $x$ an unknown scalar.

Let $\alpha_1$, $\alpha_2$, $\alpha_3$ and $g_1$, $g_2$, $g_3$ (the latter taken in descending order of magnitude), be the vector and scalar constants of $\varpi$, so that (§ 177)

$$(\varpi - g_1)\alpha_1 = 0, \text{ \&c.}$$

We have obviously, by operating on (1) with $S.\alpha_1$ &c., three equations of the form

$$S\rho\{(g_1 - x)\alpha_1 - V\epsilon\alpha_1\} = 0 \text{.......................}(2).$$

Eliminating $\rho$ (whose tensor is not involved) we have

$$S\{(g_1-x)\alpha_1 - V\epsilon\alpha_1\}\{(g_2-x)\alpha_2 - V\epsilon\alpha_2\}\{(g_3-x)\alpha_3 - V\epsilon\alpha_3\} = 0,$$

or
$$(x-g_1)(x-g_2)(x-g_3) - x\epsilon^2 + S\epsilon\varpi\epsilon = 0 \text{............}(3).$$

From each value of $x$ found from this equation the corresponding value of $\rho$ is given by (2) in the form

$$\begin{aligned}\rho &\parallel V\{(g_1-x)\alpha_1 - V\epsilon\alpha_1\}\{(g_2-x)\alpha_2 - V\epsilon\alpha_2\},\\ &\parallel (g_1-x)(g_2-x)\alpha_3 + (g_2-x)\alpha_1 S\alpha_2\epsilon - (g_1-x)\alpha_2 S\alpha_1\epsilon - \epsilon S\alpha_3\epsilon,\\ &\parallel (g_1-x)(g_2-x)\alpha_3 + V\alpha_3(\varpi - x)\epsilon - \epsilon S\alpha_3\epsilon.\end{aligned}$$

---

* *Proc. R. S. E.*, 1879—80.

The simplest method of dealing with (3) seems to be to find the limiting value of $T\epsilon$, $U\epsilon$ being given, that the roots may be all real. They are obviously real when $T\epsilon = 0$. It is clear from the properties of $\varpi$ that the extreme values of $-S\,.\,U\epsilon\,\varpi\,U\epsilon$ (which will be called $\xi$) are $g_1$ and $g_3$.

Trace the curve

$$y = (x - g_1)(x - g_2)(x - g_3),$$

and draw the (unique) tangent to it from the point $x = \xi$, $y = 0$, $\xi$ having any assigned value from $g_3$ to $g_1$. Let this tangent make an angle $-\theta$ with the axis of $x$. Suppose a simple shear to be applied to the figure so as to make this tangent turn round the point $\xi$, 0, and become the $x$ axis, while the $y$ axis is unchanged. The value of $y$ will be increased by $(x - \xi)\tan\theta$. Comparing this with (3) we see that $\tan\theta$ is the desired limiting value of $(T\epsilon)^2$.

188. We have shewn, at some length, how a linear and vector equation containing an unknown *vector* is to be solved in the most general case; and this, by § 150, shews how to find an unknown *quaternion* from any sufficiently general linear equation containing it. That such an equation may be sufficiently general it must have both scalar and vector parts: the first gives *one*, and the second *three*, scalar equations; and these are required to determine completely the four scalar elements of the unknown quaternion.

Thus $$Tq = a$$

being but one scalar equation, gives

$$q = aUr,$$

where $r$ is any quaternion whatever.

Similarly $$Sq = a$$

gives $$q = a + \theta,$$

where $\theta$ is any vector whatever. In each of these cases, only one scalar condition being given, the solution contains three scalar indeterminates. A similar remark applies to the following:

$$TVq = a$$

gives $$q = x + a\theta;$$

and $$SUq = \cos A,$$

gives $$q = x\theta^{2A/\pi},$$

in each of which $x$ is any scalar, and $\theta$ any unit vector.

189. Again, the reader may easily prove that

$$V.\alpha Vq = \beta,$$

where $\alpha$ is a given vector, gives, by putting $Sq = x$,

$$V\alpha q = \beta + x\alpha.$$

Hence, assuming $$S\alpha q = y,$$

we have $$\alpha q = y + x\alpha + \beta,$$

or $$q = x + y\alpha^{-1} + \alpha^{-1}\beta.$$

Here, the given equation being equivalent to two scalar conditions, the solution contains two scalar indeterminates.

190. Next take the equation

$$V\alpha q = \beta.$$

Operating by $S.\alpha^{-1}$, we get

$$Sq = S\alpha^{-1}\beta,$$

so that the given equation becomes

$$V\alpha(S\alpha^{-1}\beta + Vq) = \beta,$$

or $$V\alpha Vq = \beta - \alpha S\alpha^{-1}\beta = \alpha V\alpha^{-1}\beta.$$

From this, by § 170, we see that

$$Vq = \alpha^{-1}(x + \alpha V\alpha^{-1}\beta),$$

whence $$\begin{aligned} q &= S\alpha^{-1}\beta + \alpha^{-1}(x + \alpha V\alpha^{-1}\beta) \\ &= \alpha^{-1}(\beta + x), \end{aligned}$$

and, the given equation being equivalent to three scalar conditions, but one undetermined scalar remains in the value of $q$.

This solution might have been obtained at once, since our equation gives merely the *vector* of the quaternion $\alpha q$, and leaves its scalar undetermined.

Hence, taking $x$ for the scalar, we have

$$\begin{aligned} \alpha q &= S\alpha q + V\alpha q \\ &= x + \beta. \end{aligned}$$

191. Finally, of course, from

$$\alpha q = \beta,$$

which is equivalent to four scalar equations, we obtain a definite value of the unknown quaternion in the form

$$q = \alpha^{-1}\beta.$$

192. Before taking leave of linear equations, we may mention

that Hamilton has shewn how to solve any linear equation containing an unknown *quaternion*, by a process analogous to that which he employed to determine an unknown *vector* from a linear and vector equation; and to which a large part of this Chapter has been devoted. Besides the increased complexity, the peculiar feature disclosed by this beautiful discovery is that the symbolic equation for a linear quaternion function, corresponding to the cubic in $\phi$ of § 174, is a *biquadratic*, so that the inverse function is given in terms of the first, second, and third powers of the direct function. In an elementary work like the present the discussion of such a question would be out of place: although it is not very difficult to derive the more general result by an application of processes already explained. But it forms a curious example of the well-known fact that a biquadratic equation depends for its solution upon a cubic. The reader is therefore referred to the *Elements of Quaternions*, p. 491.

193. As an example of the solution of the linear equation in quaternions, let us take the problem of *finding the differential of the* n^th^ *root of a quaternion*. This comes to finding $dq$ in terms of $dr$ when

$$q^n = r.$$

[Here $n$ may obviously be treated as an integer; for, if it were fractional, both sides could be raised to the power expressed by the denominator of the fraction.]

This gives

$$q^{n-1}dq + q^{n-2}dq\,.\,q + \ldots + dq\,.\,q^{n-1} = \phi\,(dq) = dr \ldots\ldots (1),$$

and from this equation $dq$ is to be found; $\phi$ being now a *linear and quaternion* function.

Multiply *by* $q$, and then *into* $q$, and subtract. We obtain

$$q^n dq - dq\,.\,q^n = q dr - dr\,.\,q,$$

or

$$2V\,.\,Vq^nVdq = 2V\,.\,VqVdr \ldots\ldots\ldots\ldots\ldots (2).$$

But, from the equation

$$q = Sq + Vq,$$

we have at once

$$Vr = Vq^n = Q_n Vq,$$

where $Q_n = n\,(Sq)^{n-1} - \dfrac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{1\,.\,2\,.\,3}(Sq)^{n-3}\,(TVq)^2 + \&c.$

[The value of $Q_n$ is obvious from § 116, but we keep the present form.]

With this (2) becomes

$$Q_n V . VqVdq = V . VqVdr,$$

whence $$Q_n Vdq = Vdr + xVq,$$

$x$ being an undetermined scalar.

Adding another such scalar, so as to introduce $Sdq$ and $Sdr$, we have

$$Q_n dq = (y + xVq) + dr \text{..................... (3)}.$$

Substitute in (1) and we have

$$Q_n dr = nq^{n-1}(y + xVq) + \phi(dr),$$

or, by (3) again,

$$Q_n dr = nq^{n-1}(Q_n dq - dr) + \phi(dr);$$

so that, finally,

$$Q_n dq = dr + \frac{1}{n}q^{1-n}(Q_n dr - \phi(dr)) \text{.............. (4)}.$$

Thus $dq$ is completely determined.

It is interesting to form, in this case, an equation for $\phi$. This is easily done by eliminating $dr$ from (4) by the help of (1). We thus obtain

$$(\phi - nq^{n-1})(\phi - Q_n) = 0.$$

This might have been foreseen from the nature of $\phi$, as defined in (1); because it is clear that its effect on a scalar, or on a vector parallel to the axis of $q$ (which is commutative with $q$), is the same as multiplication by $nq^{n-1}$; while for any vector in the plane of $q$ it is equivalent to the scalar factor $Q_n$.

It is left to the student to solve the equation (1) by putting it in the form

$$dr = p + qpq^{-1} + q^2pq^{-2} + \ldots + q^{n-1}pq^{-n+1}$$

or $$dr - qdrq^{-1} = p - q^n pq^{-n},$$

where $$p = dqq^{n-1}.$$

The nature of the operator $q(\quad)q^{-1}$ was considered in § 119 above.

194. The question just treated involves the solution of a particular case only of the following equation:—

$$\phi(q) = \Sigma aqa' = b \text{........................ (1)},$$

where $a$, $a'$, &c. are *coplanar* quaternions.

Let $$q = r + \rho,$$

where $r$ is a quaternion coplanar with the $a$s, and $\rho$ a vector in their plane. Then, for any $a$,

$$ra = ar,$$

while
$$\rho a = Ka \,.\, \rho.$$

Thus the given equation takes the form

$$\phi(q) = \Sigma(aa') \,.\, r + \Sigma(aKa') \,.\, \rho,$$

so that the functional equation becomes in its turn

$$\{\phi - \Sigma(aa')\}\{\phi - \Sigma(aKa')\} = 0.$$

If $\alpha$ be the unit-vector perpendicular to the plane of the $a$s, we have

$$-b = \alpha S\alpha b + \alpha V\alpha b$$
$$= -(Sb - \alpha S\alpha b) + \alpha V\alpha Vb,$$

and the required solution is obviously

$$q = (\Sigma\, aa')^{-1}(Sb - \alpha S\alpha b) - (\Sigma\, aKa')^{-1}\, \alpha V\alpha Vb.$$

In the case (§ 193) of the differential of the $n^{\text{th}}$ root of a quaternion, $s$, we have

$$\Sigma(aa') = ns^{n-1},$$
$$\Sigma(aKa') = 2\,(S \,.\, s^{n-1} + T^2 s S \,.\, s^{n-3} + \ldots).$$

The last expression (in which, it must be noticed, the last term is not to be doubled when $n$ is odd) is the $Q_n$ of the former solution, though the form in which it is expressed is different. It will be a good exercise for the student to prove directly that they are equal.

195. The solution of the following frequently-occurring particular form of linear quaternion equation

$$aq + qb = c,$$

where $a$, $b$, and $c$ are any given quaternions, has been effected by Hamilton by an ingenious process, which was applied in § 140 (5) above to a simple case.

Multiply the whole *by* $Ka$, and (separately) *into* $b$, and we have

$$T^2 a \,.\, q + Ka \,.\, qb \qquad = Ka \,.\, c,$$

and
$$a \,.\, qb + qb^2 = cb.$$

Adding, we have

$$q\,(T^2 a + b^2 + 2Sa \,.\, b) = Ka \,.\, c + cb,$$

from which $q$ is at once found.

To this form any equation such as

$$a'qb' + c'qd' = e'$$

can of course be reduced, by multiplication *by* $c'^{-1}$ and *into* $b'^{-1}$.

196. To shew some of the characteristic peculiarities in the solution of quaternion equations even of the first degree when they are not sufficiently general, let us take the very simple one

$$aq = qb,$$

and give every step of the solution, as practice in transformations.

Apply Hamilton's process (§ 195), and we get

$$T^2a \,.\, q = Ka \,.\, qb,$$

$$qb^2 = aqb.$$

These give $$q\,(T^2a + b^2 - 2bSa) = 0,$$

so that the equation gives no real finite value for $q$ unless

$$T^2a + b^2 - 2bSa = 0,$$

or $$b = Sa + \beta TVa,$$

where $\beta$ is some unit-vector. This gives $Sa = Sb$.

By a similar process we may evidently shew that

$$a = Sb + \alpha TVb,$$

$\alpha$ being another unit-vector.

But, by the given equation,

$$Ta = Tb,$$

or $$S^2a + T^2Va = S^2b + T^2Vb\,;$$

from which, and the above values of $a$ and $b$, we see that we may write

$$\frac{Sa}{TVa} = \frac{Sb}{TVb} = \mathrm{a}, \text{ suppose.}$$

Thus we may write

$$a = \mathrm{a} + \alpha,\ b = \mathrm{a} + \beta,$$

where $\alpha$ and $\beta$ are unit-vectors.

If, then, we separate $q$ into its scalar and vector parts, thus

$$q = u + \rho,$$

the given equation becomes

$$(\mathrm{a} + \alpha)(u + \rho) = (u + \rho)(\mathrm{a} + \beta) \;\ldots\ldots\ldots\ldots\ldots(1).$$

Multiplying out we have

$$u\,(\alpha - \beta) = \rho\beta - \alpha\rho,$$

which gives $$S\,(\alpha - \beta)\,\rho = 0,$$

and therefore $$\rho = V\gamma\,(\alpha - \beta),$$

where $\gamma$ is an undetermined vector.

We have now

$$\begin{aligned} u(\alpha-\beta) &= \rho\beta-\alpha\rho \\ &= V\gamma(\alpha-\beta).\beta-\alpha V\gamma(\alpha-\beta) \\ &= \gamma(S\alpha\beta+1)-(\alpha-\beta)S\beta\gamma-\gamma(1+S\alpha\beta)-(\alpha-\beta)S\alpha\gamma \\ &= -(\alpha-\beta)S(\alpha+\beta)\gamma. \end{aligned}$$

Having thus determined $u$, we have

$$\begin{aligned} q &= -S(\alpha+\beta)\gamma+V\gamma(\alpha-\beta), \\ 2q &= -(\alpha+\beta)\gamma-\gamma(\alpha+\beta)+\gamma(\alpha-\beta)-(\alpha-\beta)\gamma \\ &= -2\alpha\gamma-2\gamma\beta. \end{aligned}$$

Here, of course, we may change the sign of $\gamma$, and write the solution of

$$aq = qb$$

in the form $$q=\alpha\gamma+\gamma\beta,$$

where $\gamma$ is any vector, and

$$\alpha = UVa, \qquad \beta = UVb.$$

To verify this solution, we see by (1) that we require only to shew that

$$\alpha q = q\beta.$$

But their common value is evidently

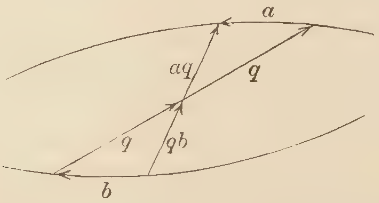
$$-\gamma+\alpha\gamma\beta.$$

An *apparent* increase of generality of this solution may be obtained by writing

$$q=\alpha r+r\beta$$

where $r$ is any *quaternion*. But this is easily seen to be equivalent to adding to $\gamma$ (which is *any vector*) a term of the form $xV\alpha\beta$.

It will be excellent practice for the student to represent the terms of this equation by versor-arcs, as in § 54, and to deduce the above solution from the diagram annexed:—



The vector of the intersection of the plane of $q$, with that of $aq$

and $qb$, is evidently symmetrically situated with regard to the great circles of $a$ and $b$. Hence it is parallel to

$$(\alpha+\beta)\, V\alpha\beta, \text{ i.e. to } \alpha-\beta.$$

Let $\gamma$ be any vector in the plane of $a$.

Then
$$q \propto \gamma(\beta-\alpha),$$
$$\propto \alpha\gamma+\gamma\beta,$$

because $S\alpha\gamma=0$, and thus $-\gamma\alpha=\alpha\gamma$.

Another simple form of solution consists in writing the equation as

$$a=qbq^{-1},$$

and applying the results of § 119.

197. No general quaternion method of solving equations of the second or higher degrees has yet been found; in fact, as will be shewn immediately, even those of the second degree involve (in their most general form) algebraic equations of the *sixteenth* degree. Hence, in the few remaining sections of this Chapter we shall confine ourselves to one or two of the simpler forms for the treatment of which a definite process has been devised. But first, let us consider how many roots an equation of the second degree in an unknown quaternion must generally have.

If we substitute for the quaternion the expression

$$w+ix+jy+kz \text{ (§ 80)},$$

and treat the quaternion constants in the same way, we shall have on development (§ 80) four equations, generally of the second degree, to determine $w, x, y, z$. The number of roots will therefore be $2^4$ or 16. And similar reasoning shews us that a quaternion equation of the $m$th degree has $m^4$ roots. It is easy to see, however, from some of the simple examples given above (§§ 188—190, &c.) that, unless the given equation is equivalent to four independent scalar equations, the roots will contain one or more indeterminate quantities.

198. Hamilton has effected in a simple way the solution of the quadratic

$$q^2=qa+b,$$

or the following, which is virtually the same (as we see by taking the conjugate of each side),

$$q^2=aq+b.$$

He puts $q = \frac{1}{2}(a + w + \rho)$,
where $w$ is a scalar, and $\rho$ a vector.

Substituting this value in the first written form of the equation, we get

$$a^2 + (w+\rho)^2 + 2wa + a\rho + \rho a = 2(a^2 + wa + \rho a) + 4b,$$

or
$$(w+\rho)^2 + a\rho - \rho a = a^2 + 4b.$$

If we put $Va = \alpha$, $S(a^2 + 4b) = c$, $V(a^2 + 4b) = 2\gamma$, this becomes

$$(w+\rho)^2 + 2V\alpha\rho = c + 2\gamma;$$

which, by equating separately the scalar and vector parts, may be broken up into the two equations

$$w^2 + \rho^2 = c,$$

$$V(w+\alpha)\rho = \gamma.$$

The latter of these can be solved for $\rho$ by the process of § 168; or more simply by operating at once by $S.\alpha$, which gives the value of $S(w+\alpha)\rho$. If we substitute the resulting value of $\rho$ in the former we obtain, as the reader may easily prove, the equation

$$(w^2 - \alpha^2)(w^4 - cw^2 + \gamma^2) - (V\alpha\gamma)^2 = 0.$$

The solution of this scalar cubic gives six values of $w$, for each of which we find a value of $\rho$, and thence a value of $q$.

Hamilton shews (*Lectures*, p. 633) that only two of these values are real quaternions, the remaining four being biquaternions, and the other ten roots of the given equation being infinite.

Hamilton farther remarks that the above process leads, as the reader may easily see, to the solution of the two simultaneous equations

$$q + r = a,$$

$$qr = -b;$$

and he connects it also with the evaluation of certain continued fractions with quaternion constituents. (See the Miscellaneous Examples at the end of this volume.)

199. The equation $q^2 = aq + qb$,

though apparently of the second degree, is easily reduced to the first degree by multiplying *by*, and *into*, $q^{-1}$, when it becomes

$$1 = q^{-1}a + bq^{-1},$$

and may be treated by the process of § 195.

The equation

$$V.q(\alpha + V\beta q) = 0,$$

where $\alpha$ and $\beta$ are given vectors, is easily seen to require for a *real* (i.e. a non biquaternion) solution that $q$ shall be a vector. Hence we may write it as

$$V.\rho(\alpha+V\beta\rho)=0;$$

whence, at once,

$$\alpha+V\beta\rho=x\rho.$$

Assume $$S\beta\rho=y,$$

and we have

$$-(y-\alpha)=(x-\beta)\rho,$$

or $$-(x+\beta)(y-\alpha)=(x^2-\beta^2)\rho.$$

The condition that $\rho$ is a vector gives

$$xy-S\alpha\beta=0,$$

so that the value of $\rho$, containing one scalar indeterminate, is

$$\rho=-(x-\beta)^{-1}\left(\frac{S\alpha\beta}{x}-\alpha\right).$$

To determine $\rho$ completely we require one additional scalar condition.

If we have, for instance,

$$S\gamma\rho=e,$$

$x$ is given by the cubic equation

$$(x^2-\beta^2)e=xS\alpha\gamma-\frac{S\alpha\beta S\beta\gamma}{x}-S.\alpha\beta\gamma.$$

But if the condition be that $\rho$ is a vector-radius of the unit sphere (a result which will be required below) we have the biquadratic

$$x^2-\beta^2=\frac{S^2\alpha\beta}{x^2}-\alpha^2.$$

This gives two real values of $x^2$, but they have opposite signs; so that there are always two, and only two, real values of $x$.

200. The equation $$q^m=aqb,$$

where $a$ and $b$ are given quaternions, gives

$$q(aqb)=(aqb)q;$$

and, by § 54, it is evident that the planes of $q$ and $aqb$ must coincide. A little consideration (after the manner of the latter part of § 196) will shew that the solution depends upon drawing two arcs which shall intercept given arcs upon each of two great circles; while one of them bisects the other, and is divided by it in the proportion of $m:1$. The equation treated in § 196 is the special case of this when $m=1$.

## EXAMPLES TO CHAPTER V.

1. Solve the following equations :—

(*a*) $V.\alpha\rho\beta = V.\alpha\gamma\beta.$

(*b*) $\alpha\rho\beta\rho = \rho\alpha\rho\beta.$

(*c*) $\alpha\rho + \rho\beta = \gamma.$

(*d*) $S.\alpha\beta\rho + \beta S\alpha\rho - \alpha V\beta\rho = \gamma.$

(*e*) $\rho + \alpha\rho\beta = \alpha\beta.$

(*f*) $\alpha\rho\beta\rho = \rho\beta\rho\alpha.$

Do any of these impose any restriction on the generality of $\alpha$ and $\beta$?

2. Suppose $\rho = ix + jy + kz,$

and $-\phi\rho = aiSi\rho + bjSj\rho + ckSk\rho\,;$

put into Cartesian cöordinates the following equations :—

(*a*) $T\phi\rho = 1.$

(*b*) $S\rho\phi^2\rho = -1.$

(*c*) $S.\rho(\phi^2 + \rho^2)^{-1}\rho = -1.$

(*d*) $T\rho = T.\phi U\rho.$

3. If $\lambda$, $\mu$, $\nu$ be *any* three non-coplanar vectors, and

$$q = V\mu\nu\,.\,\phi\lambda + V\nu\lambda\,.\,\phi\mu + V\lambda\mu\,.\,\phi\nu,$$

shew that $q$ is necessarily divisible by $S.\lambda\mu\nu$.

Also shew that the quotient is

$$m_2 - 2\epsilon,$$

where $V\epsilon\rho$ is the non-commutative part of $\phi\rho$.

Hamilton, *Elements*, p. 442.

4. Solve the simultaneous equations :—

(*a*) $\left.\begin{aligned} S\alpha\rho &= 0, \\ S.\alpha\rho\phi\rho &= 0. \end{aligned}\right\}$

(*b*) $\left.\begin{aligned} S\alpha\rho &= 0, \\ S\rho\phi\rho &= 0. \end{aligned}\right\}$

(*c*) $\left.\begin{aligned} S\alpha\rho &= 0, \\ S.\alpha\iota\rho\kappa\rho &= 0. \end{aligned}\right\}$

5. If $$\phi\rho = \Sigma\beta S\alpha\rho + Vr\rho,$$
where $r$ is a given quaternion, shew that
$$m = \Sigma(S.\alpha_1\alpha_2\alpha_3 S.\beta_3\beta_2\beta_1) + \Sigma S(rV\alpha_1\alpha_2 . V\beta_2\beta_1) + Sr\Sigma S.\alpha\beta r$$
$$- \Sigma(S\alpha r S\beta r) + SrTr^2,$$
and $$m\phi^{-1}\sigma = \Sigma(V\alpha_1\alpha_2 S.\beta_2\beta_1\sigma) + \Sigma V.\alpha V(V\beta\sigma.r) + V\sigma r Sr - VrS\sigma r.$$
*Lectures*, p. 561.

6. If $[pq]$ denote $pq - qp$,
$(pqr)$ „ $S.p[qr]$,
$[pqr]$ „ $(pqr) + [rq]Sp + [pr]Sq + [qp]Sr$,
and $(pqrs)$ „ $S.p[qrs]$;
shew that the following relations exist among any five quaternions
$$0 = p(qrst) + q(rstp) + r(stpq) + s(tpqr) + t(pqrs),$$
and $$q(prst) = [rst]Spq - [stp]Srq + [tpr]Ssq - [prs]Stq.$$
*Elements*, p. 492.

7. Shew that if $\phi$, $\psi$ be any linear and vector functions, and $\alpha$, $\beta$, $\gamma$ rectangular unit-vectors, the vector
$$\theta = V(\phi\alpha\psi\alpha + \phi\beta\psi\beta + \phi\gamma\psi\gamma)$$
is an invariant. [This will be immediately seen if we write it in the form $$\theta = V.\phi\nabla\psi\rho,$$
which is independent of the directions of $\alpha$, $\beta$, $\gamma$. But it is good practice to dispense with $\nabla$, when possible.]

If $$\phi\rho = \Sigma\eta S\zeta\rho,$$
and $$\psi\rho = \Sigma\eta_1 S\zeta_1\rho,$$
shew that this invariant may be expressed as
$$-\Sigma V\eta\psi\zeta \text{ or } \Sigma V\eta_1\phi\zeta_1.$$
Shew also that $$\phi\psi'\rho - \psi\phi'\rho = -V\theta\rho.$$
The scalar of the same quaternion is also an invariant, and may be written as
$$-\Sigma\Sigma_1 S\eta\eta_1 S\zeta\zeta_1$$
$$= -\Sigma S\eta\psi\zeta$$
$$= -\Sigma_1 S\eta_1\phi\zeta_1.$$

8. Shew that if $$\phi\rho = \alpha S\alpha\rho + \beta S\beta\rho + \gamma S\gamma\rho,$$
where $\alpha$, $\beta$, $\gamma$ are any three vectors, then
$$\phi^{-1}\rho S^2.\alpha\beta\gamma = \alpha_1 S\alpha_1\rho + \beta_1 S\beta_1\rho + \gamma_1 S\gamma_1\rho,$$
where $\alpha_1 = V\beta\gamma$, &c.

9. Shew that any self-conjugate linear and vector function may in general be expressed in terms of two given ones, the expression involving terms of the second order.

Shew also that we may write

$$\phi + z = a(\varpi + x)^2 + b(\varpi + x)(\omega + y) + c(\omega + y)^2,$$

where $a, b, c, x, y, z$ are scalars, and $\varpi$ and $\omega$ the two given functions. What character of generality is necessary in $\varpi$ and $\omega$? How is the solution affected by non-self-conjugation in one or both?

10. Solve the equations :—

$$\begin{array}{ll} (a) & q^2 = 5qi + 10j. \\ (b) & q^2 = 2q + i. \\ (c) & qaq = bq + c. \\ (d) & aq = qr = rb. \end{array}$$

11. Shew that $\phi V\nabla\phi\rho = -mV\nabla\phi^{-1}\rho.$

12. If $\phi$ be self-conjugate, and $\alpha, \beta, \gamma$ a rectangular system,

$$S \,.\, V\alpha\phi\alpha V\beta\phi\beta V\gamma\phi\gamma = 0.$$

13. $\phi\psi$ and $\psi\phi$ give the same values of the invariants $m$, $m_1$, $m_2$.

14. If $\phi'$ be conjugate to $\phi$, $\phi\phi'$ is self-conjugate.

15. Shew that $(V\alpha\theta)^2 + (V\beta\theta)^2 + (V\gamma\theta)^2 = 2\theta^2$ if $\alpha, \beta, \gamma$ be rectangular unit-vectors.

16. Prove that $\nabla^2(\phi - g)\rho = -\rho\nabla^2 g + 2\nabla g.$

17. Solve the equations :—

$$(a)\quad \phi^2 = \varpi;$$

$$(b)\quad \left.\begin{array}{r} \phi + \chi = \varpi, \\ \phi\chi = \theta; \end{array}\right\}$$

where one, or two, unknown linear and vector functions are given in terms of known ones. (Tait, *Proc. R. S. E.* 1870–71.)

18. If $\phi$ be a self-conjugate linear and vector function, $\xi$ and $\eta$ two vectors, the two following equations are consequences one of the other, viz. :—

$$\frac{\xi}{S^{\frac{1}{3}} \,.\, \xi\phi\xi\phi^2\xi} = \frac{V \,.\, \eta\phi\eta}{S^{\frac{2}{3}} \,.\, \eta\phi\eta\phi^2\eta},$$

$$\frac{\eta}{S^{\frac{1}{3}} \,.\, \eta\phi\eta\phi^2\eta} = \frac{V\xi\phi\xi}{S^{\frac{2}{3}} \,.\, \xi\phi\xi\phi^2\xi}.$$

From either of them we obtain the equation

$$S\phi\xi\phi\eta = S^{\frac{1}{3}}.\,\xi\phi\xi\phi^2\xi\, S^{\frac{1}{3}}.\,\eta\phi\eta\phi^2\eta.$$

This, taken along with one of the others, gives a singular theorem when translated into ordinary algebra. What property does it give of the surface

$$S.\,\rho\phi\rho\phi^2\rho = 1\,?$$ [*Ibid.*]

19. Solve the equation

$$q\alpha q = \beta q\beta.$$

Shew that it has a vector solution, involving the trisection of an angle:—and find the condition that it shall admit of a real quaternion solution.

20. Solve

$$bqaq = qbqa,$$

and state the corresponding geometrical problem: shewing that when $a$ and $b$ are equal *vectors*, $q$ is equal to each.

21. Given $\phi$, a self-conjugate linear and vector function, and a vector $\epsilon$; find the cubic in $\psi$, where

$$\psi\rho = \phi\rho + V\epsilon\rho.$$

22. Investigate the simplest expressions for *any* linear and vector function in terms of given ones:—and point out what degree of generality is necessary in the latter.

Why cannot the conjugate of a linear and vector function be generally expressed in powers of the function itself?

# CHAPTER VI.

## SKETCH OF THE ANALYTICAL THEORY OF QUATERNIONS.

(By Prof. Cayley.)

### (*a*) *Expression, Addition, and Multiplication.*

By what precedes we are led to an analytical theory of the Quaternion $q = w + ix + jy + kz$, where the imaginary symbols $i, j, k$ are such that

$$i^2 = -1, j^2 = -1, k^2 = -1, jk = -kj = i, ki = -ik = j, ij = -ji = k.$$

The Tensor $Tq$ is $= \sqrt{w^2 + x^2 + y^2 + z^2}$, and

the Versor $Uq$ is $= \dfrac{1}{\sqrt{w^2 + x^2 + y^2 + z^2}}(w + ix + jy + kz)$,

which, or the quaternion itself when $Tq = 1$, may be expressed in the form

$$\cos \delta + \sin \delta \, (ia + jb + kc) \text{ where } a^2 + b^2 + c^2 = 1;$$

such a quaternion is a Unit Quaternion. The squared tensor $w^2 + x^2 + y^2 + z^2$ is called the Norm.

The scalar part $Sq$ is $= w$, and the vector part $Vq$, or say a Vector, is $= ix + jy + kz$. The Length is $= \sqrt{x^2 + y^2 + z^2}$, and the quotient $\dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(ix + jy + kz)$, or say a vector $ix + jy + kz$ where $x^2 + y^2 + z^2 = 1$, is a Unit Vector.

The quaternions $w + ix + jy + kz$ and $w - ix - jy - kz$ are said to be Conjugates, each of the other. Conjugate quaternions have the same norm; and the product of the conjugate quaternions is the norm of either of them. The conjugate of a quaternion is denoted by $\bar{q}$, or $Kq$.

Quaternions $q = w + ix + jy + kz$, $q' = w' + ix' + jy' + kz'$ are added by the formula

$$q + q' = w + w' + i(x + x') + j(y + y') + k(z + z'),$$

the operation being commutative and associative.

They are multiplied by the formula

$$\begin{aligned} qq' = \quad & ww' - xx' - yy' - zz' \\ & + i(wx' + xw' + yz' - zy') \\ & + j(wy' + yw' + zx' - z'x) \\ & + k(wz' + zw' + xy' - x'y), \end{aligned}$$

where observe that the norm is

$$= (w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2)$$

the product of the norms of $q$ and $q'$.

The multiplication is *not* commutative, $q'q \neq qq'$; but it *is* associative, $qq' \,.\, q'' = q \,.\, q'q'' = qq'q''$, &c. In combination with addition it is distributive, $q(q' + q'') = qq' + qq''$, &c.

### (*b*) *Imaginary Quaternions. Nullitats.*

The components $w$, $x$, $y$, $z$ of a quaternion are usually real, but they may be imaginary of the form $a + b\sqrt{-1}$, where $\sqrt{-1}$ is the imaginary of ordinary algebra: we cannot (as in ordinary algebra) represent this by the letter $i$, but when occasion requires another letter, say $\theta$, may be adopted (the meaning, $\theta = \sqrt{-1}$, being explained). An imaginary quaternion is thus a quaternion of the form $(w + \theta w_1) + i(x + \theta x_1) + j(y + \theta y_1) + k(z + \theta z_1)$, or, what is the same thing, if $q$, $q_1$ be the real quaternions $w + ix + jy + kz$, $w_1 + ix_1 + jy_1 + kz_1$, it is a quaternion $q + \theta q_1$; this algebraical imaginary $\theta = \sqrt{-1}$ is commutative with each of the symbols $i, j, k$: or, what comes to the same thing, it is not in general necessary to explicitly introduce $\theta$ at all, but we work with the quaternion $w + ix + jy + kz$, in exactly the same way as if $w, x, y, z$ were real values. A quaternion of the above form, $q + \theta q_1$, was termed by Hamilton a "biquaternion" but it seems preferable to speak of it simply as a quaternion, using the term biquaternion only for a like expression $q + \theta q_1$, wherein $\theta$ is not the $\sqrt{-1}$ of ordinary algebra.

It may be noticed that, for an imaginary quaternion, the squared tensor or norm $w^2 + x^2 + y^2 + z^2$ may be $= 0$; when this is so, the quaternion is said to be a "Nullitat"; the case is one to be separately considered.

### (c) *Quaternion as a Matrix.*

Quaternions have an intimate connection with Matrices. Suppose that $\theta, = \sqrt{-1}$, is the $\sqrt{-1}$ of ordinary algebra, and in place of $i, j, k$ consider the new imaginaries $x, y, z, w$ which are such that

$$\begin{aligned} x &= \tfrac{1}{2}(\ \ 1 - \theta i), \text{ or conversely } & 1 &= \ \ x + w, \\ y &= \tfrac{1}{2}(\ \ j - \theta k), & i &= \theta(x - w), \\ z &= \tfrac{1}{2}(-j - \theta k), & j &= \ \ (y - z), \\ w &= \tfrac{1}{2}(\ \ 1 + \theta i), & k &= \theta(y + z); \end{aligned}$$

so that $a, b, c, d$ being scalars, $ax + by + cz + dw$ denotes the imaginary quaternion

$$\tfrac{1}{2}\{a + d + (b - c)j\} + \tfrac{1}{2}\{(-a + d)i + (-b - c)k\}\theta.$$

We obtain for $x, y, z, w$ the laws of combination

$$\begin{array}{c|cccc|} & x & y & z & w \\ \hline x & x & y & 0 & 0 \\ y & 0 & 0 & x & y \\ z & z & w & 0 & 0 \\ w & 0 & 0 & z & w \end{array}\ , \text{ that is } x^2 = x,\ xy = y,\ xz = 0,\ xw = 0 \ \ \&c.,$$

and consequently for the product of two linear forms in $(x, y, z, w)$ we have

$$\begin{aligned} &(ax + by + cz + dw)(a'x + b'y + c'z + d'w) \\ &\qquad = (aa' + bc')x + (ab' + bd')y + (ca' + dc')z + (cb' + dd')w; \end{aligned}$$

and this is precisely the form for the product of two matrices, viz. we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = \begin{array}{c|cc} & (a', c') & (b', d') \\ \hline (a, b) & ,, & ,, \\ (c, d) & ,, & ,, \end{array} = \begin{vmatrix} aa' + bc', & ab' + bd' \\ ca' + dc', & cb' + dd' \end{vmatrix}$$

and hence the linear form $ax + by + cz + dw$, and the matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ may be regarded as equivalent symbols. This identification was established by the remark and footnote "Peirce's Linear Associative Algebra," *Amer. Math. Jour.* t. 4 (1881), p. 132.

### (d) *The Quaternion Equation* $\Sigma AqB = C$.

In ordinary algebra, an equation of the first degree, or linear equation with one unknown quantity $x$, is merely an equation of the form $ax = b$, and it gives at once $x = a^{-1}b$.

But the case is very different with quaternions; the general form of a linear equation with one unknown quaternion $q$ is

$$A_1qB_1 + A_2qB_2 + \ldots = C, \text{ or say } \Sigma AqB = C,$$

where $C$ and the several coefficients $A$ and $B$ are given quaternions.

Considering the expression on the left-hand side, and assuming $q = w + ix + jy + kz$, it is obvious that the expression is in effect of the form

$$\begin{aligned} &\delta w + \alpha x + \beta y + \gamma z \\ &+ i(\delta_1 w + \alpha_1 x + \beta_1 y + \gamma_1 z) \\ &+ j(\delta_2 w + \alpha_2 x + \beta_2 y + \gamma_2 z) \\ &+ k(\delta_3 w + \alpha_3 x + \beta_3 y + \gamma_3 z), \end{aligned}$$

where the coefficients $\delta$, $\alpha$, $\beta$, $\gamma$ &c. are given scalar magnitudes: if then this is equal to a given quaternion $C$, say this is

$$\lambda + i\lambda_1 + j\lambda_2 + k\lambda_3,$$

we have for the determination of $w$, $x$, $y$, $z$ the four equations

$$\begin{aligned} \delta w + \alpha x + \beta y + \gamma z &= \lambda, \\ \delta_1 w + \alpha_1 x + \beta_1 y + \gamma_1 z &= \lambda_1, \\ \delta_2 w + \alpha_2 x + \beta_2 y + \gamma_2 z &= \lambda_2, \\ \delta_3 w + \alpha_3 x + \beta_3 y + \gamma_3 z &= \lambda_3, \end{aligned}$$

and we thence have $w$, $x$, $y$, $z$, each of them as a fraction with a given numerator, and with the common denominator

$$\Delta = \begin{vmatrix} \delta, & \alpha, & \beta, & \gamma \\ \delta_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \delta_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \delta_3, & \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix},$$

viz. this is the determinant formed with the coefficients $\delta$, $\alpha$, $\beta$, $\gamma$, &c. Of course if $\Delta = 0$, then either the equations are inconsistent, or they reduce themselves to fewer than four independent equations.

The number of these coefficients is $= 16$, and it is thus clear that, whatever be the number of the terms $A_1qB_1$, $A_2qB_2$, &c. we only in effect introduce into the equation 16 coefficients. A single term such as $A_1qB_1$ may be regarded as containing seven coefficients, for we may without loss of generality write it in the form

$$g(1 + ia + jb + kc)\, q\, (1 + id + je + kf),$$

and thus we do not obtain the general form of linear equation by taking a single term $A_1qB_1$ (for this contains seven coefficients only) nor by taking two terms $A_1qB_1$, $A_2qB_2$ (for these contain 14 coefficients only); but we do, it would seem, obtain the general form by taking three terms (viz. these contain 21 coefficients, which must in effect reduce themselves to 16): that is, a form $A_1qB_1 + A_2qB_2 + A_3qB_3$ is, or seems to be, capable of representing the above written quaternion form with any values whatever of the 16 coefficients $\delta$, $\alpha$, $\beta$, $\gamma$ &c. But the further theory of this reduction to 16 coefficients is not here considered.

The most simple case of course is that of a single term, say we have $AqB = C$: here multiplying on the left by $A^{-1}$ and on the right by $B^{-1}$, we obtain at once $q = A^{-1}CB^{-1}$.

### (e) *The Nivellator, and its Matrix.*

In the general case, a solution, equivalent to the foregoing, but differing from it very much in form may be obtained by means of the following considerations.

A symbol of the above form $\Sigma A\,(\quad)\,B$, operating upon a quaternion $q$ so as to change it into $\Sigma A\,(q)\,B$, is termed by Prof. Sylvester a "Nivellator:" it may be represented by a single letter, say we have $\phi = \Sigma A\,(\quad)\,B$; the effect of it, as has just been seen, is to convert the components $(w, x, y, z)$, into four linear functions $(w_1, x_1, y_1, z_1)$ which may be expressed by the equation

$$(w_1, x_1, y_1, z_1) = \left|\begin{array}{cccc} \delta, & \alpha, & \beta, & \gamma \\ \delta_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \delta_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \delta_3, & \alpha_3, & \beta_3, & \gamma_3 \end{array}\right| (w, x, y, z),$$

or say by the multiplication of $(w, x, y, z)$ by a matrix which may be called the matrix of the nivellator; and the theory of the solution of the linear equation in quaternions thus enters into relation with that of the solution of the linear equation in matrices.

The operation denoted by $\phi$ admits of repetition: we have for instance

$$\{A_1(\quad)B_1 + A_2(\quad)B_2\}^2 = A_1^2(\quad)B_1^2 + A_1A_2(\quad)B_2B_1 + A_2A_1(\quad)B_1B_2 + A_2^2(\quad)B_2^2;$$

and similarly for more than two terms, and for higher powers.

Considering $\phi$ in connexion with its matrix $M$, we have $M^2(w, x, y, z)$ for the components of $\phi^2(q)$, $M^3(w, x, y, z)$ for those of $\phi^3(q)$, and so on. Hence also we have the negative powers $\phi^{-1}$, &c. of the operation $\phi$. The mode in which $\phi^{-1}$ can be calculated will presently appear: but assuming for the moment that it can be calculated, the given equation is $\phi(q) = C$, that is we have $q = \phi^{-1}(C)$, the solution of the equation.

A matrix $M$ of any order satisfies identically an equation of the same order: viz. for the foregoing matrix $M$ of the fourth order we have

$$\begin{vmatrix} \delta - M, & \alpha, & \beta, & \gamma \\ \delta_1, & \alpha_1 - M, & \beta_1, & \gamma_1 \\ \delta_2, & \alpha_2, & \beta_2 - M, & \gamma_2 \\ \delta_3, & \alpha_3, & \beta_3, & \gamma_3 - M \end{vmatrix}$$

viz. this is

$$M^4 - eM^3 + fM^2 - gM + h = 0,$$

where $h$ is the before mentioned determinant

$$\begin{vmatrix} \delta, & \alpha, & \beta, & \gamma \\ \delta_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \delta_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \delta_3, & \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix}, \text{ say this is, } h = \Delta.$$

$M$, in its operation on the components $(w, x, y, z)$ of $q$, exactly represents $\phi$ in its operation on $q$: we thus have

$$\phi^4 - e\phi^3 + f\phi^2 - g\phi + h = 0,$$

viz. this means that operating successively with $\phi$ on the arbitrary quaternion $Q$ we have identically

$$\phi^4(Q) - e\phi^3(Q) + f\phi^2(Q) - g\phi(Q) + hQ = 0;$$

where observe that the coefficients $e, f, g, h$ have their foregoing values, calculated by means of the minors of the determinant: but that their values may also be calculated quite independently of this determinant: viz. the equation shews that there is an identical linear relation connecting the values $\phi^4(Q)$, $\phi^3(Q)$, $\phi^2(Q)$, $\phi(Q)$ and $Q$: and from the values (assumed to be known) of these quantities, we can calculate the identical equation which connects them. But in whatever way they are found, the coefficients $e, f, g, h$ are to be regarded as known scalar functions.

Writing in the equation $\phi^{-1}Q$ in place of $Q$, we have

$$\phi^3(Q) - e\phi^2(Q) + f\phi(Q) - gQ + h\phi^{-1}(Q) = 0,$$

viz. this equation gives $\phi^{-1}(Q)$ as a linear function of $Q, \phi(Q), \phi^2(Q)$ and $\phi^3(Q)$: and hence for the arbitrary quaternion $Q$ writing the value $C$, we have $q, = \phi^{-1}(C)$ given as a linear function of

$C$, $\phi(C)$, $\phi^2(C)$ and $\phi^3(C)$: we have thus the solution of the given linear equation.

### ($f$) *The Vector Equation* $\Sigma A\rho B = C$.

The theory is similar if, instead of quaternions, we have vectors. As to this observe in the first place that, even if $A$, $q$, $B$ are each of them a vector, the product $AqB$ will be in general, not a vector, but a quaternion. Hence in the equation $\Sigma AqB = C$, if $C$ and the several coefficients $A$ and $B$ be all of them vectors, the quantity $q$ as determined by this equation will be in general a quaternion: and even if it should come out to be a vector, still in the process of solution it will be necessary to take account, not only of the vector components, but also of the scalar part; so that there is here no simplification of the foregoing general theory.

But the several coefficients $A$, $B$ may be vectors so related to each other that the sum $\Sigma A\rho B$, where $\rho$ is an arbitrary vector, is always a vector[1]; and in this case, if $C$ be also a vector, the equation $\Sigma A\rho B = C$ will determine $\rho$ as a vector: and there is here a material simplification. Writing $\rho = ix + jy + kz$, then $\Sigma A\rho B$ is in effect of the form

$$\begin{aligned} &i(\alpha_1 x + \beta_1 y + \gamma_1 z) \\ &+j(\alpha_2 x + \beta_2 y + \gamma_2 z) \\ &+k(\alpha_3 x + \beta_3 y + \gamma_3 z), \end{aligned}$$

viz. we have these three linear functions of $(x, y, z)$ to be equalled to given scalar values $\lambda_1$, $\lambda_2$, $\lambda_3$, and here $x$, $y$, $z$ have to be determined by the solution of the three linear equations thus obtained. And for the second form of solution, writing as before $\phi = \Sigma A(\quad)B$, then $\phi$ is connected with the more simple matrix

$$M = \left|\begin{matrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{matrix}\right|$$

and it thus (instead of a biquadratic equation) satisfies the cubic equation

$$\left|\begin{matrix} \alpha_1 - M, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2 - M, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 - M \end{matrix}\right| = 0,$$

say

$$M^3 - eM^2 + fM - g = 0.$$

---

[1] Thus, if $A$, $B$ are conjugate quaternions, $A\rho B$ is a vector $\sigma$: this is in fact the form which presents itself in the theory of rotation.

We have therefore for $\phi$ the cubic equation

$$\phi^3 - e\phi^2 + f\phi - g = 0,$$

and thus $\phi^{-1}(Q)$ is given as a linear function of $Q$, $\phi(Q)$, $\phi^2(Q)$, or, what is the same thing, $\phi^{-1}(C)$ as a linear function of $C$, $\phi(C)$, $\phi^2(C)$: and (this being so) then for the solution of the given equation $\phi(\rho) = C$, we have $\rho = \phi^{-1}(C)$, a given linear function of $C$, $\phi(C)$, $\phi^2(C)$.

### (*g*) *Nullitats.*

Simplifications and specialities present themselves in particular cases, for instance in the cases $Aq + qB = C$, and $Aq = qB$, which are afterwards considered.

The product of a quaternion into its conjugate is equal to the squared tensor, or norm; $aa = T^2(a)$; and thus the reciprocal of a quaternion is equal to the conjugate divided by the norm; hence if the norm be $= 0$, or say if the quaternion be a nullitat, there is no reciprocal. In particular, 0, quà quaternion, is a nullitat.

The equation $aqb = c$, where $a$, $b$, $c$ are given quaternions, $q$ the quaternion sought for, is at once solvable; we have $q = a^{-1}cb^{-1}$; but the solution fails if $a$, or $b$, or each of them, is a nullitat. And when this is so, then whatever be the value of $q$, we have $aqb$ a nullitat, and thus the equation has no solution unless also $c$ be a nullitat.

If $a$ and $c$ are nullitats, but $b$ is not a nullitat, then the equation gives $aq = cb^{-1}$, which is of the form $aq = c$; and similarly if $b$ and $c$ are nullitats but $a$ is not a nullitat, then the equation gives $qb = a^{-1}c$, which is of the form $qb = c$: thus the forms to be considered are $aq = c$, $qb = c$, and $aqb = c$, where in the first equation $a$ and $c$, in the second equation $b$ and $c$, and in the third equation $a$, $b$, $c$, are nullitats.

The equation $aq = c$, $a$ and $c$ nullitats, does not in general admit of solution, but when it does so, the solution is indeterminate; viz. if $Q$ be a solution, then $Q + aR$ (where $R$ is an arbitrary quaternion) is also a solution. Similarly for the equation $qb = c$, if $Q$ be a solution, then $Q + Sb$ ($S$ an arbitrary quaternion) is also a solution: and in like manner for the equation $aqb = c$, if $Q$ be a solution then also $Q + aR + Sb$ ($R$, $S$ arbitrary quaternions) is a solution.

### (*h*) *Conditions of Consistency, when some Coefficients are Nullitats.*

Consider first the equation $aq=c$; writing $a=a_4+ia_1+ja_2+ka_3$ $(a_4^2+a_1^2+a_2^2+a_3^2=0)$ and $c=c_4+ic_1+jc_2+kc_3\,(c_4^2+c_1^2+c_2^2+c_3^2=0)$; also $q=w+ix+jy+kz$, the equation gives

$$\begin{aligned} c_4 &= a_4w - a_1x - a_2y - a_3z, \\ c_1 &= a_1w + a_4x - a_3y + a_2z, \\ c_2 &= a_2w + a_3x + a_4y - a_1z, \\ c_3 &= a_3w - a_2x + a_1y + a_4z, \end{aligned}$$

equations which are only consistent with each other when two of the $c$'s are determinate linear functions of the other two $c$'s; and when this is so, the equations reduce themselves to *two* independent equations. Thus from the first, second and third equations, multiplying by $a_1a_3-a_2a_4, -a_4a_3-a_1a_2$, and $-a_2^2-a_3^2$, and adding, we obtain

$$(a_1a_3-a_2a_4)\,c_4-(a_3a_4+a_1a_2)\,c_1-(a_2^2+a_3^2)\,c_2=0\,;$$

similarly from the first, second and fourth equations, multiplying by $-a_3a_4-a_1a_2, -a_1a_3+a_2a_4, -a_2^2-a_3^2$, and adding, we have

$$-(a_1a_2+a_3a_4)\,c_4-(a_1a_3-a_2a_4)\,c_1-(a_2^2+a_3^2)\,c_3=0,$$

and when these two equations are satisfied, the original equations are equivalent to two independent equations; so that we have for instance a solution $Q=w+ix$ where $c_4=a_4w-a_1x$, $c_1=a_1w+a_4x$, that is $w=\dfrac{a_4c_4+a_1c_1}{a_4^2+a_1^2}$, $x=\dfrac{-a_1c_4+a_4c_1}{a_4^2+a_1^2}$; and the general solution is then obtained as above.

The equations connecting the $a$'s and the $c$'s may be presented in a variety of different forms, all of them of course equivalent in virtue of the relations $a_4^2+a_1^2+a_2^2+a_3^2=0$, $c_4^2+c_1^2+c_2^2+c_3^2=0$; viz. writing

$$\begin{aligned} A_1&=a_1^2+a_4^2=-a_2^2-a_3^2, & F_1&=a_2a_3+a_1a_4, & F_1'&=a_2a_3-a_1a_4, \\ A_2&=a_2^2+a_4^2=-a_3^2-a_1^2, & F_2&=a_3a_1+a_2a_4, & F_2'&=a_3a_1-a_2a_4, \\ A_3&=a_3^2+a_4^2=-a_1^2-a_2^2, & F_3&=a_1a_2+a_3a_4, & F_3'&=a_1a_2-a_3a_4, \end{aligned}$$

then the relation between any three of the $c$'s may be expressed in three different forms, with coefficients out of the sets $A_1, A_2, A_3$; $F_1, F_2, F_3$; $F_1', F_2', F_3'$. Obviously the relation between the $c$'s is satisfied if $c=0$: the equation then is $aq=0$, satisfied by $q=aR$, $R$ an arbitrary quaternion.

We have a precisely similar theory for the equation $qb = c$; any two of the $c$'s must be determinate linear functions of the other two of them; and we have then only two independent equations for the determination of the $w$, $x$, $y$, $z$.

In the case of the equation $aqb = c$ ($a$, $b$, $c$ all nullitats) the analysis is somewhat more complicated, but the final result is a simple and remarkable one; from the condition that $a$, $b$ are nullitats, it follows that $ab$, $aib$, $ajb$, $akb$ are scalar (in general imaginary scalar) multiples of one and the same nullitat, say of $ab$: the condition to be satisfied by $c$ then is that $c$ shall be a scalar multiple of this same nullitat, say $c = \lambda ab$; the equation $aqb = \lambda ab$ has then a solution $q = \lambda$, and the general solution is $q = \lambda + \bar{a}R + Sb$, where $R$, $S$ are arbitrary quaternions.

## (i) *The Linear Equations*, $aq - qb = 0$, *and* $aq - qb = c$.

The foregoing considerations explain a point which presents itself in regard to the equation $aq - qb = 0$, ($a$, $b$ given quaternions, $q$ a quaternion sought for): clearly the equation is not solvable (otherwise than by the value $q = 0$) unless a condition be satisfied by the given quaternions $a$, $b$; but this condition is *not* (what at first sight it would appear to be) $T^2a = T^2b$. The condition (say $\Omega$) may be satisfied although $T^2a \neq T^2b$; and being satisfied, there exists a determinate quaternion $q$, which must evidently be a nullitat (for from the given equation $aq = qb$ we have $(T^2a - T^2b)\,T^2q = 0$, that is $T^2q = 0$). If in addition to the condition $\Omega$ we have also $T^2a - T^2b = 0$, then (as will appear) we have an indeterminate solution $q$, which is not in general a nullitat.

Take the more general equation $aq - qb = c$: this may be solved by a process (due to Hamilton) as follows: multiplying on the left hand by $\bar{a}$ and on the right hand by $b$, we have $\bar{a}aq - \bar{a}qb = \bar{a}c$, $aqb - qb^2 = cb$, whence subtracting

$$a\bar{a}q - (a + \bar{a})\,qb + qb^2 = \bar{a}c - cb,$$

or since $a\bar{a}$, $a + \bar{a}$ are scalars $q\,\{a\bar{a} - (a + a)\,b + b^2\} = ac - cb$: viz. this is an equation of the form $qB = C$ ($B$, $C$ given quaternions), having a solution $q = CB^{-1}$.

Suppose $c = 0$, then also $C = 0$; and unless $B$ is a nullitat, the equation $qB = 0$ (representing the original equation $aq = qb$), has only the solution $q = 0$; viz. the condition in order that the equation $aq=qb$ may have a solution other than $q=0$, is $B=$ nullitat, that is $aa-(a+\bar{a})\,b+b^2=$ nullitat; viz. we must have

$$\begin{aligned}&a_4^2+a_1^2+a_2^2+a_3^2\\&-2a_4(b_4+ib_1+jb_2+kb_3)\\&+b_4^2+2b_4\,(ib_1+jb_2+kb_3)-b_1^2-b_2^2-b_3^2=\text{nullitat},\end{aligned}$$

that is

$$\begin{aligned}a_4^2+a_1^2+a_2^2+a_3^2-2a_4b_4+b_4^2-b_1^2-b_2^2-b_3^2&\\+2\,(b_4-a_4)\,(ib_1+jb_2+kb_3)&=\text{nullitat}.\end{aligned}$$

The condition $\Omega$ thus is

$$(a_4^2+a_1^2+a_2^2+a_3^2-2a_4b_4+b_4^2-b_1^2-b_2^2-b_3^2)^2+4(a_4-b_4)^2(b_1^2+b_2^2+b_3^2)=0,$$

that is

$$\{(a_4-b_4)^2+a_1^2+a_2^2+a_3^2-b_1^2-b_2^2-b_3^2\}^2+4(a_4-b_4)^2(b_1^2+b_2^2+b_3^2)=0,$$

or, as this may also be written,

$$\begin{aligned}(a_4-b_4)^4+2\,(a_4-b_4)^2\,(a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2)&\\+(a_1^2+a_2^2+a_3^2-b_1^2-b_2^2-b_3^2)^2&=0.\end{aligned}$$

Writing herein

$$a_4^2+a_1^2+a_2^2+a_3^2=A^2,\quad b_4^2+b_1^2+b_2^2+b_3^2=B^2,$$

the condition is

$$(a_4-b_4)^4+2\,(a_4-b_4)^2\,(A^2+B^2-a_4^2-b_4^2)+(A^2-B^2-a_4^2+b_4^2)^2=0,$$

which is easily reduced to

$$4\,(a_4-b_4)\,(a_4B^2-b_4A^2)+(A^2-B^2)^2=0,$$

and, as already noticed, this is different from $T^2a-T^2b=0$, that is $A^2-B^2=0$.

If the equation $A^2-B^2=0$ is satisfied, then the condition $\Omega$ reduces itself to $a_4-b_4=0$; we then have $a=a_4+\alpha$, $b=a_4+\beta$, where $\alpha$, $\beta$ are vectors, and the equation is therefore $\alpha q=q\beta$ where (since $A^2-B^2, =a_1^2+a_2^2+a_3^2-b_1^2-b_2^2-b_3^2, =0$), the tensors are equal, or we may without loss of generality take $\alpha$, $\beta$ to be given unit vectors, viz. we have $\alpha^2=-1$, $\beta^2=-1$: and this being so, we obtain at once the solution $q=\lambda\,(\alpha+\beta)+\mu\,(1-\alpha\beta)$ ($\lambda$, $\mu$, arbitrary scalars): in fact this value gives

$$\alpha q=\lambda\,(-1+\alpha\beta)+\mu\,(\alpha+\beta)=q\beta.$$

Reverting to the general equation $aq-qb=c$, the conjugate of $a\bar{a}-(a+\bar{a})\,b+b^2$ is $a\bar{a}-(a+\bar{a})\,\bar{b}+\bar{b}^2$, and we thus obtain the solution

$$q\,\{4\,(a_4-b_4)\,(a_4B^2-b_4A^2)+(A^2-B^2)^2\}=(\bar{a}c-cb)\,\{a\bar{a}-(a+\bar{a})\,\bar{b}+\bar{b}^2\},$$

but this solution fails if $a\bar{a} - (a + \bar{a})\, b + b^2$ is a nullitat: supposing it to be so, the equation is only solvable when $C$ satisfies the condition which expresses that the equation $qB = C$ is solvable when $B$, $C$ are nullitats.

The equation $aq - qb = c$, could it is clear be in like manner reduced to the form $Aq = C$.

### (*j*) *The Quadric Equation* $q^2 - 2aq + b = 0$.

We consider the quadric equation $q^2 - 2aq + b = 0$; $a$ and $b$ given quaternions, $q$ the quaternion sought for. The solution which follows is that given by Prof. Sylvester for a quadric equation in binary matrices.

In general if $q$ be any quaternion, $= w + ix + jy + kz$, then $(q - w)^2 + x^2 + y^2 + z^2 = 0$, that is $q^2 - 2qw + w^2 + x^2 + y^2 + z^2 = 0$, or say $q^2 - 2q\,(\text{scal. } q) + \text{norm } q = 0$: viz. this is an identical relation connecting a quaternion with its scalar and its norm.

Writing as above $q = w + ix + jy + kz$, and $t = w^2 + x^2 + y^2 + z^2$ for the norm, we thus have

$$q^2 - 2wq + t = 0,$$

and combining this with the given equation

$$q^2 - 2aq + b = 0,$$

we find $2\,(a - w)\, q - (b - t) = 0$, that is $2q = (a - w)^{-1}\,(b - t)$,

an expression for $q$ in terms of the scalar and norm $w$, $t$, and of the known quaternions $a$ and $b$.

$2q$ as thus determined satisfies the identical equation

$$(2q)^2 - 2\,(2q)\,\text{scal. } \{(a - w)^{-1}\,(b - t)\} + \text{norm } \{(a - w)^{-1}\,(b - t)\} = 0,$$

and we have

$$\text{scal. } \{(a - w)^{-1}\,(b - t)\} = \frac{\text{scal. } \{(\bar{a} - w)\,(b - t)\}}{\text{norm } (a - w)},$$

$$\text{norm } \{(a - w)^{-1}\,(b - t)\} = \frac{\text{norm } (b - t)}{\text{norm } (a - w)},$$

($\bar{a}$ the conjugate of $a$).

The equation thus becomes

$$4q^2\,\text{norm } (a - w) - 4q\,\{\text{scal. } (\bar{a} - w)\,(b - t)\} + \text{norm } (b - t) = 0:$$

this must agree with

$$q^2 \qquad\qquad - 2qw \qquad\qquad\qquad + \qquad\qquad t = 0,$$

or say the function is $= 4\lambda\,(q^2 - 2qw + t)$; we thus have

$$\begin{aligned} \text{norm}\,(a-w) &= \lambda, \\ \text{scal.}\,(\bar{a}-w)\,(b-t) &= 2\lambda w, \\ \text{norm}\,(b-t) &= 4\lambda t, \end{aligned}$$

three equations for the determination of $\lambda$, $w$, $t$; and then, $w$, $t$ being determined, the required value of $q$ is $2q = (a-w)^{-1}\,(b-t)$ as above.

To develope the solution let the values of a, b, c, f, g, h be defined as follows: viz.

$$\text{norm}\,(ax + by + z) = (\mathrm{a, b, c, f, g, h} \between x, y, z)^2,$$

viz. writing

$$\begin{aligned} a &= a_4 + ia_1 + ja_2 + ka_3, \\ b &= b_4 + ib_1 + jb_2 + kb_3, \end{aligned}$$

then this equation is

$$(a_4x + b_4y + z)^2 + (a_1x + b_1y)^2 + (a_2x + b_2y)^2 + (a_3x + b_3y)^2 \\ = (\mathrm{a, b, c, f, g, h} \between x, y, z)^2,$$

that is, a, b, c, f, g, h denote as follows

$$\begin{aligned} \mathrm{a} &= a_4^2 + a_1^2 + a_2^2 + a_3^2, & \mathrm{f} &= b_4, \\ \mathrm{b} &= b_4^2 + b_1^2 + b_2^2 + b_3^2, & \mathrm{g} &= a_4, \\ \mathrm{c} &= 1, & \mathrm{h} &= a_4b_4 + a_1b_1 + a_2b_2 + a_3b_3. \end{aligned}$$

We then have

$$\begin{aligned} \text{norm}\,(a-w) &= (a_4-w)^2 + a_1^2 + a_2^2 + a_3^2, \\ \text{scal.}\,(\bar{a}-w)\,(b-t) &= (a_4-w)\,(b_4-t) + a_1b_1 + a_2b_2 + a_3b_3, \\ \text{norm}\,(b-t) &= (b_4-t)^2 + b_1^2 + b_2^2 + b_3^2, \end{aligned}$$

or expressing these in terms of (a, b, c, f, g, h) the foregoing three equations become

$$\begin{aligned} \mathrm{a} - 2\mathrm{g}w + \mathrm{c}w^2 &= \lambda, \\ \mathrm{h} - \mathrm{g}t - \mathrm{f}w + \mathrm{c}tw &= 2\lambda w, \\ \mathrm{b} - 2\mathrm{f}t + \mathrm{c}t^2 &= 4\lambda t, \end{aligned}$$

where c (introduced only for greater symmetry) is $= 1$.

Writing moreover $A, B, C, F, G, H = \mathrm{bc} - \mathrm{f}^2,\ \mathrm{ca} - \mathrm{g}^2,\ \mathrm{ab} - \mathrm{h}^2,\ \mathrm{gh} - \mathrm{af},\ \mathrm{hf} - \mathrm{bg},\ \mathrm{fg} - \mathrm{ch}$, and $K = \mathrm{abc} - \mathrm{af}^2 - \mathrm{bg}^2 - \mathrm{ch}^2 + 2\mathrm{fgh}$; also in place of $w$, $t$ introducing into the equations $u = w - \mathrm{g}$, and $v = t - \mathrm{f}$, the equations become

$$\begin{aligned} u^2 + B &= \lambda, \\ uv - H &= 2\lambda\,(u + \mathrm{g}), \\ v^2 + A &= 4\lambda\,(v + \mathrm{f}). \end{aligned}$$

We deduce
$$u^2 = \lambda - B,$$
$$u\,(v - 2\lambda) = H + 2\lambda \mathrm{g},$$
$$(v - 2\lambda)^2 = 4\lambda^2 + 4\lambda \mathrm{f} - A,$$
and we thence obtain, to determine $\lambda$, the cubic equation
$$(\lambda - B)\,(4\lambda^2 + 4\lambda \mathrm{f} - A) - (2\lambda \mathrm{g} + H)^2 = 0,$$
viz. this is
$$4\lambda^3 + 4\lambda^2\,(\mathrm{f} - \mathrm{a}) + \lambda\,\{-\mathrm{bc} + \mathrm{f}^2 + 4\,(\mathrm{gh} - \mathrm{af})\}$$
$$+ \mathrm{c}\,(\mathrm{abc} - \mathrm{af}^2 - \mathrm{bg}^2 - \mathrm{ch}^2 + 2\mathrm{fgh}) = 0,$$
that is,
$$4\lambda^3 + 4\lambda^2\,(\mathrm{f} - \mathrm{a}) + \lambda\,(-A + 4F) + K = 0,$$
and, $\lambda$ being determined by this equation, then
$$u = \pm\sqrt{\lambda - B}, \quad v = 2\lambda + \frac{H + 2\lambda g}{u},$$
and then $w = u + \mathrm{g}$, $t = v + \mathrm{f}$; consequently
$$2q = (a - \mathrm{g} - u)^{-1}\,(b - \mathrm{f} - v).$$

Write for a moment $a - \mathrm{g} - u = \Theta$, then
$$\Theta\,(\Theta + 2u) = (a - \mathrm{g})^2 - u^2 = a^2 - 2a\mathrm{g} + \mathrm{a} - B - u^2, \ = -\lambda$$
(since $a = \mathrm{g} + ia_1 + ja_2 + ka_3$, $\mathrm{a} = \mathrm{g}^2 + a_1^2 + a_2^2 + a_3^2$ and thus the identical equation for $a$ is $a^2 - 2a\mathrm{g} + \mathrm{a} = 0$): that is $\Theta^2 + 2u\Theta + \lambda = 0$, or $\lambda\Theta^{-1} = -(\Theta + 2u) = (a - \mathrm{g} + u)$; that is $\Theta^{-1}$, $= (a - \mathrm{g} - u)^{-1}$, $= -\dfrac{1}{\lambda}(a - \mathrm{g} + u)$; and the value of $q$ is $2q = -\dfrac{1}{\lambda}(a - \mathrm{g} + u)\,(b - \mathrm{f} - v)$, or say it is
$$2q = -\frac{1}{\lambda}\,(a - \mathrm{g} + u)\left(b - \mathrm{f} - 2\lambda - \frac{H + 2\lambda \mathrm{g}}{u}\right),$$
where $\lambda$ is determined by the cubic equation, and $u$ is $= \pm\sqrt{\lambda - B}$; we have thus six roots of the given quadric equation $q^2 - 2aq + b = 0$.

# CHAPTER VII.

## GEOMETRY OF THE STRAIGHT LINE AND PLANE.

201. HAVING, in the preceding Chapters, given a brief exposition of the theory and properties of quaternions, we intend to devote the rest of the work to examples of their practical application, commencing, of course, with the simplest curve and surface, the straight line and the plane. In this and the remaining Chapters of the work a few of the earlier examples will be wrought out in their fullest detail, with a reference to the previous part of the book whenever a transformation occurs; but, as each Chapter proceeds, superfluous steps will be gradually omitted, until in the later examples the full value of the quaternion processes is exhibited.

202. Before proceeding to the proper business of the Chapter we make a digression in order to give a few instances of applications to ordinary plane geometry. These the student may multiply indefinitely with great ease.

(*a*) *Euclid*, I. 5. Let $\alpha$ and $\beta$ be the vector sides of an isosceles triangle; $\beta - \alpha$ is the base, and

$$T\alpha = T\beta.$$

The proposition will evidently be proved if we shew that

$$\alpha (\alpha - \beta)^{-1} = K\beta (\beta - \alpha)^{-1} \quad (\S\ 52).$$

This gives $\alpha (\alpha - \beta)^{-1} = (\beta - \alpha)^{-1} \beta,$

or $(\beta - \alpha)\, \alpha = \beta (\alpha - \beta),$

or $-\alpha^2 = -\beta^2.$

(*b*) *Euclid*, I. 32. Let $ABC$ be the triangle, and let

$$U\frac{\overline{AC}}{\overline{AB}} = \gamma^{l},$$

where $\gamma$ is a unit-vector perpendicular to the plane of the triangle. If $l = 1$, the angle $CAB$ is a right angle (§ 74). Hence

$A = l\,\pi/2$ (§ 74). Let $B = m\,\pi/2$, $C = n\,\pi/2$. We have

$$U\overline{AC} = \gamma^l U\overline{AB},$$

$$U\overline{CB} = \gamma^n U\overline{CA},$$

$$U\overline{BA} = \gamma^m U\overline{BC}.$$

Hence $$U\overline{BA} = \gamma^m \,.\, \gamma^n \,.\, \gamma^l U\overline{AB},$$

or $$-1 = \gamma^{l+m+n}.$$

That is $$l + m + n = 2,$$

or $$A + B + C = \pi.$$

This is, properly speaking, Legendre's proof; and might have been given in a far shorter form than that above. In fact we have for any three vectors whatever,

$$U\,.\,\frac{\alpha}{\beta}\,\frac{\beta}{\gamma}\,\frac{\gamma}{\alpha} = 1,$$

which contains Euclid's proposition as a mere particular case.

(*c*) *Euclid*, I. 35. Let $\beta$ be the common vector-base of the parallelograms, $\alpha$ the conterminous vector-side of any one of them. For any other the vector-side is $\alpha + x\beta$ (§ 28), and the proposition appears as

$$TV\beta\,(\alpha + x\beta) = TV\beta\alpha \text{ (§§ 96, 98)},$$

which is obviously true.

(*d*) In the base of a triangle find the point from which lines, drawn parallel to the sides and limited by them, are equal.

If $\alpha$, $\beta$ be the sides, any point in the base has the vector

$$\rho = (1 - x)\,\alpha + x\beta.$$

For the required point

$$(1 - x)\,T\alpha = xT\beta$$

which determines $x$.

Hence the point lies on the line

$$\rho = y\,(U\alpha + U\beta)$$

which bisects the vertical angle of the triangle.

This is not the only solution, for we should have written

$$T(1 - x)\,T\alpha = TxT\beta,$$

instead of the less general form above *which tacitly assumes that* $1 - x$ *and* $x$ *have the same sign*. We leave this to the student.

(*e*) If perpendiculars be erected outwards at the middle points of the sides of a triangle, each being proportional to the corresponding side, the mean point of the triangle formed by their extremities coincides with that of the original triangle. Find the ratio of each perpendicular to half the corresponding side of the old triangle that the new triangle may be equilateral.

Let $2\alpha$, $2\beta$, and $2(\alpha+\beta)$ be the vector-sides of the triangle, $i$ a unit-vector perpendicular to its plane, $e$ the ratio in question. The vectors of the corners of the new triangle are (taking the corner opposite to $2\beta$ as origin)

$$\rho_1 = \alpha + ei\alpha,$$

$$\rho_2 = 2\alpha + \beta + ei\beta,$$

$$\rho_3 = \alpha + \beta - ei(\alpha + \beta).$$

From these

$$\tfrac{1}{3}(\rho_1 + \rho_2 + \rho_3) = \tfrac{1}{3}(4\alpha + 2\beta) = \tfrac{1}{3}\{2\alpha + 2(\alpha + \beta)\},$$

which proves the first part of the proposition.

For the second part, we must have

$$T(\rho_2 - \rho_1) = T(\rho_3 - \rho_2) = T(\rho_1 - \rho_3).$$

Substituting, expanding, and erasing terms common to all, the student will easily find

$$3e^2 = 1.$$

Hence, if equilateral triangles be described on the sides of any triangle, their mean points form an equilateral triangle.

203. Such applications of quaternions as those just made are of course legitimate, but they are not always profitable. In fact, when applied to plane problems, quaternions often degenerate into mere scalars, and become (§ 33) Cartesian cöordinates of some kind, so that nothing is gained (though nothing is lost) by their use. Before leaving this class of questions we take, as an additional example, the investigation of some properties of the ellipse.

204. We have already seen (§ 31 (*k*)) that the equation

$$\rho = \alpha\cos\theta + \beta\sin\theta$$

represents an ellipse, $\theta$ being a scalar which may have any value. Hence, for the vector-tangent at the extremity of $\rho$ we have

$$\varpi = \frac{d\rho}{d\theta} = -\alpha\sin\theta + \beta\cos\theta,$$

which is easily seen to be the value of $\rho$ when $\theta$ is increased by $\pi/2$. Thus it appears that any two values of $\rho$, for which $\theta$ differs by $\pi/2$, are conjugate diameters. The area of the parallelogram circumscribed to the ellipse and touching it at the extremities of these diameters is, therefore, by § 96,

$$4TV\rho\frac{d\rho}{d\theta} = 4TV(\alpha\cos\theta + \beta\sin\theta)(-\alpha\sin\theta + \beta\cos\theta)$$

$$= 4TV\alpha\beta,$$

a constant, as is well known.

205. For equal conjugate diameters we must have

$$T(\alpha\cos\theta + \beta\sin\theta) = T(-\alpha\sin\theta + \beta\cos\theta),$$

or
$$(\alpha^2 - \beta^2)(\cos^2\theta - \sin^2\theta) + 4S\alpha\beta\cos\theta\sin\theta = 0,$$

or
$$\tan 2\theta = -\frac{\alpha^2 - \beta^2}{2S\alpha\beta}.$$

The square of the common length of these diameters is of course

$$-\frac{\alpha^2 + \beta^2}{2},$$

because we see at once from § 204 that the sum of the squares of conjugate diameters is constant.

206. The maximum or minimum of $\rho$ is thus found;

$$\frac{dT\rho}{d\theta} = -\frac{1}{T\rho}S\rho\frac{d\rho}{d\theta},$$

$$= -\frac{1}{T\rho}\{-(\alpha^2 - \beta^2)\cos\theta\sin\theta + S\alpha\beta(\cos^2\theta - \sin^2\theta)\}.$$

For a maximum or minimum this must vanish*, hence

$$\tan 2\theta = \frac{2S\alpha\beta}{\alpha^2 - \beta^2},$$

and therefore the longest and shortest diameters are equally inclined to each of the equal conjugate diameters (§ 205). Hence, also, they are at right angles to each other.

207. Suppose for a moment $\alpha$ and $\beta$ to be the greatest and least semidiameters, so that

$$S\alpha\beta = 0.$$

---

* The student must carefully notice that here we put $\frac{dT\rho}{d\theta} = 0$, and not $\frac{d\rho}{d\theta} = 0$. A little reflection will shew him that the latter equation involves an absurdity.

Then the equations of any two tangent-lines are

$$\rho = \alpha\cos\theta + \beta\sin\theta + x\,(-\alpha\sin\theta + \beta\cos\theta),$$

$$\rho = \alpha\cos\theta_1 + \beta\sin\theta_1 + x_1(-\alpha\sin\theta_1 + \beta\cos\theta_1).$$

If these tangent-lines be at right angles to each other

$$S(-\alpha\sin\theta + \beta\cos\theta)(-\alpha\sin\theta_1 + \beta\cos\theta_1) = 0$$

or $$\alpha^2\sin\theta\sin\theta_1 + \beta^2\cos\theta\cos\theta_1 = 0.$$

Also, for their point of intersection we have, by comparing coefficients of $\alpha$, $\beta$ in the above values of $\rho$,

$$\cos\theta - x\sin\theta = \cos\theta_1 - x_1\sin\theta_1,$$

$$\sin\theta + x\cos\theta = \sin\theta_1 + x_1\cos\theta_1.$$

Determining $x_1$ from these equations, we easily find

$$T\rho^2 = -(\alpha^2 + \beta^2),$$

the equation of a circle; if we take account of the above relation between $\theta$ and $\theta_1$.

Also, as the equations above give $x = -x_1$, the tangents are equal multiples of the diameters parallel to them; so that the line joining the points of contact is parallel to that joining the extremities of these diameters.

208. Finally, when the tangents

$$\rho = \alpha\cos\theta + \beta\sin\theta + x\,(-\alpha\sin\theta + \beta\cos\theta),$$

$$\rho = \alpha\cos\theta_1 + \beta\sin\theta_1 + x_1(-\alpha\sin\theta_1 + \beta\cos\theta_1),$$

meet in a given point

$$\rho = a\alpha + b\beta,$$

we have $$a = \cos\theta - x\sin\theta = \cos\theta_1 - x_1\sin\theta_1,$$

$$b = \sin\theta + x\cos\theta = \sin\theta_1 + x_1\cos\theta_1.$$

Hence $$x^2 = a^2 + b^2 - 1 = x_1^{\,2}$$

and $$a\cos\theta + b\sin\theta = 1 = a\cos\theta_1 + b\sin\theta_1$$

determine the values of $\theta$ and $x$ for the directions and lengths of the two tangents. The equation of the chord of contact is

$$\rho = y\,(\alpha\cos\theta + \beta\sin\theta) + (1-y)(\alpha\cos\theta_1 + \beta\sin\theta_1).$$

If this pass through the point

$$\rho = p\alpha + q\beta,$$

we have $$p = y\cos\theta + (1-y)\cos\theta_1,$$

$$q = y\sin\theta + (1-y)\sin\theta_1,$$

from which, by the equations which determine $\theta$ and $\theta_1$, we get

$$ap + bq = y + 1 - y = 1.$$

Thus if either $a$ and $b$, or $p$ and $q$, be given, a linear relation connects the others. This, by § 30, gives all the ordinary properties of poles and polars.

209. Although, in §§ 28—30, we have already given some of the equations of the line and plane, these were adduced merely for their applications to anharmonic coördinates and transversals; and not for investigations of a higher order. Now that we are prepared to determine the lengths and inclinations of lines we may investigate these and other similar forms anew.

210. *The equation of the indefinite line drawn through the origin $O$, of which the vector $\overline{OA}$, $=\alpha$, forms a part,* is evidently

$$\rho = x\alpha,$$

or

$$\rho \parallel \alpha,$$

or

$$V\alpha\rho = 0,$$

or

$$U\rho = U\alpha;$$

the essential characteristic of these equations being that they are linear, and involve *one* indeterminate scalar in the value of $\rho$.

We may put this perhaps more clearly if we take any two vectors, $\beta$, $\gamma$, which, along with $\alpha$, form a non-coplanar system. Operating with $S.V\alpha\beta$ and $S.V\alpha\gamma$ upon any of the preceding equations (except the third, and on it by $S.\beta$ and $S.\gamma$) we get

and

$$\left.\begin{aligned} S.\alpha\beta\rho &= 0 \\ S.\alpha\gamma\rho &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Separately, these are the equations of the planes containing $\alpha$, $\beta$, and $\alpha$, $\gamma$; together, of course, they denote the line of intersection.

211. Conversely, to solve equations (1), or to find $\rho$ in terms of known quantities, we see that they may be written

$$\left.\begin{aligned} S.\rho V\alpha\beta &= 0 \\ S.\rho V\alpha\gamma &= 0 \end{aligned}\right\},$$

so that $\rho$ is perpendicular to $V\alpha\beta$ and $V\alpha\gamma$, and is therefore parallel to the vector of their product. That is,

$$\rho \parallel V.V\alpha\beta V\alpha\gamma,$$

$$\parallel -\alpha S.\alpha\beta\gamma,$$

or

$$\rho = x\alpha.$$

212. By putting $\rho-\beta$ for $\rho$ we change the origin to a point $B$ where $\overline{OB}=-\beta$, or $\overline{BO}=\beta$; so that the equation of a line parallel to $\alpha$, and passing through the extremity of a vector $\beta$ drawn from the origin, is

$$\rho-\beta=x\alpha,$$

or

$$\rho=\beta+x\alpha.$$

Of course any two parallel lines may be represented as

$$\rho=\beta+x\alpha,$$

$$\rho=\beta_1+x_1\alpha;$$

or

$$V\alpha(\rho-\beta)=0,$$

$$V\alpha(\rho-\beta_1)=0.$$

213. *The equation of a line, drawn through the extremity of* $\beta$, *and meeting* $\alpha$ *perpendicularly*, is thus found. Suppose it to be parallel to $\gamma$, its equation is

$$\rho=\beta+x\gamma.$$

To determine $\gamma$ we know, *first*, that it is perpendicular to $\alpha$, which gives

$$S\alpha\gamma=0.$$

*Secondly*, $\alpha$, $\beta$, and $\gamma$ are in one plane, which gives

$$S.\alpha\beta\gamma=0.$$

These two equations give

$$\gamma \parallel V.\alpha V\alpha\beta,$$

whence we have

$$\rho=\beta+x\alpha V\alpha\beta.$$

This might have been obtained in many other ways; for instance, we see at once that

$$\beta=\alpha^{-1}\alpha\beta=\alpha^{-1}S\alpha\beta+\alpha^{-1}V\alpha\beta.$$

This shews that $\alpha^{-1}V\alpha\beta$ (which is evidently perpendicular to $\alpha$) is coplanar with $\alpha$ and $\beta$, and is therefore the direction of the required line; so that its equation is

$$\rho=\beta+y\alpha^{-1}V\alpha\beta,$$

the same as before if we put $-\dfrac{y}{T\alpha^2}$ for $x$.

214. By means of the last investigation we see that

$$-\alpha^{-1}V\alpha\beta$$

is the vector perpendicular drawn from the extremity of $\beta$ to the line

$$\rho=x\alpha.$$

Changing the origin, we see that

$$-\alpha^{-1}V\alpha(\beta-\gamma)$$

is the vector perpendicular from the extremity of $\beta$ upon the line

$$\rho=\gamma+x\alpha.$$

215. The vector joining $B$ (where $\overline{OB}=\beta$) with any point in

$$\rho=\gamma+x\alpha$$

is

$$\gamma+x\alpha-\beta.$$

Its length is least when

$$dT(\gamma+x\alpha-\beta)=0,$$

or

$$S\alpha(\gamma+x\alpha-\beta)=0,$$

i.e. when it is perpendicular to $\alpha$.

The last equation gives

$$x\alpha^2+S\alpha(\gamma-\beta)=0,$$

or

$$x\alpha=-\alpha^{-1}S\alpha(\gamma-\beta).$$

Hence the vector perpendicular is

$$\gamma-\beta-\alpha^{-1}S\alpha(\gamma-\beta),$$

or

$$\alpha^{-1}V\alpha(\gamma-\beta)=-\alpha^{-1}V\alpha(\beta-\gamma),$$

which agrees with the result of last section.

216. *To find the shortest vector distance between two lines in space*

$$\rho=\beta+x\alpha,$$

and

$$\rho_1=\beta_1+x_1\alpha_1;$$

we must put

$$dT(\rho-\rho_1)=0,$$

or

$$S(\rho-\rho_1)(d\rho-d\rho_1)=0,$$

or

$$S(\rho-\rho_1)(\alpha dx-\alpha_1 dx_1)=0.$$

Since $x$ and $x_1$ are independent, this breaks up into the two conditions

$$S\alpha(\rho-\rho_1)=0,$$

$$S\alpha_1(\rho-\rho_1)=0;$$

proving the well-known truth that the required line is perpendicular to each of the given lines.

Hence it is parallel to $V\alpha\alpha_1$, and therefore we have

$$\rho-\rho_1=\beta+x\alpha-\beta_1-x_1\alpha_1=yV\alpha\alpha_1\text{............}(1).$$

Operate by $S.\alpha\alpha_1$ and we get

$$S.\alpha\alpha_1(\beta-\beta_1)=y(V\alpha\alpha_1)^2.$$

This determines $y$, and the shortest distance required is

$$T(\rho-\rho_1)=T(yV\alpha\alpha_1)=\frac{TS.\alpha\alpha_1(\beta-\beta_1)}{TV\alpha\alpha_1}=TS.(UV\alpha\alpha_1)(\beta-\beta_1).$$

[*Note.* In the two last expressions $T$ before $S$ is inserted simply to ensure that the length be taken positively. If

$$S.\alpha\alpha_1(\beta-\beta_1) \text{ be negative,}$$

then (§ 89) $\quad S.\alpha_1\alpha(\beta-\beta_1)$ is positive.

If we omit the $T$, we must use in the text that one of these two expressions which is positive.]

To find the extremities of this shortest distance, we must operate on (1) with $S.\alpha$ and $S.\alpha_1$. We thus obtain two equations, which determine $x$ and $x_1$, as $y$ is already known.

A somewhat different mode of treating this problem will be discussed presently.

217. *In a given tetrahedron to find a set of rectangular cöordinate axes, such that each axis shall pass through a pair of opposite edges.*

Let $\alpha$, $\beta$, $\gamma$ be three (vector) edges of the tetrahedron, one corner being the origin. Let $\rho$ be the vector of the origin of the sought rectangular system, which may be called $i$, $j$, $k$ (unknown vectors). The condition that $i$, drawn from $\rho$, intersects $\alpha$ is

$$S.i\alpha\rho=0 \quad\text{.............................}(1).$$

That it intersects the opposite edge, whose equation is

$$\varpi=\beta+x(\beta-\gamma),$$

the condition is

$$S.i(\beta-\gamma)(\rho-\beta)=0, \text{ or } Si\{(\beta-\gamma)\rho-\beta\gamma\}=0 \quad ...(2).$$

There are two other equations like (1), and two like (2), which can be at once written down.

Put
$$\beta-\gamma=\alpha_1,\quad \gamma-\alpha=\beta_1,\quad \alpha-\beta=\gamma_1,$$
$$V\beta\gamma=\alpha_2,\quad V\gamma\alpha=\beta_2,\quad V\alpha\beta=\gamma_2,$$
$$V\alpha_1\alpha=\alpha_3,\quad V\beta_1\beta=\beta_3,\quad V\gamma_1\gamma=\gamma_3;$$

and the six become

$$S.i\alpha\rho=0,\qquad S.i\alpha_1\rho-Si\alpha_2=0,$$
$$S.j\beta\rho=0,\qquad S.j\beta_1\rho-Sj\beta_2=0,$$
$$S.k\gamma\rho=0,\qquad S.k\gamma_1\rho-Sk\gamma_2=0.$$

The two in $i$ give $\quad i\parallel\alpha S\alpha_2\rho-\rho(S\alpha\alpha_2+S\alpha_3\rho).$

Similarly,

$$j \parallel \beta S\beta_2\rho - \rho(S\beta\beta_2 + S\beta_3\rho), \text{ and } k \parallel \gamma S\gamma_2\rho - \rho(S\gamma\gamma_2 + S\gamma_3\rho).$$

The conditions of rectangularity, viz.,

$$Sij = 0, \quad Sjk = 0, \quad Ski = 0,$$

at once give three equations of the fourth order, the first of which is

$$0 = S\alpha\beta S\alpha_2\rho S\beta_2\rho - S\alpha\rho S\alpha_2\rho(S\beta\beta_2 + S\beta_3\rho) - S\beta\rho S\beta_2\rho(S\alpha\alpha_2 + S\alpha_3\rho)$$
$$+ \rho^2(S\alpha\alpha_2 + S\alpha_3\rho)(S\beta\beta_2 + S\beta_3\rho).$$

The required origin of the rectangular system is thus given as the intersection of three surfaces of the fourth order.

218. The equation $S\alpha\rho = 0$

imposes on $\rho$ the sole condition of being perpendicular to $\alpha$; and therefore, being satisfied by the vector drawn from the origin to any point in a plane through the origin and perpendicular to $\alpha$, is the equation of that plane.

To find this equation by a direct process similar to that usually employed in cöordinate geometry, we may remark that, by § 29, we may write

$$\rho = x\beta + y\gamma,$$

where $\beta$ and $\gamma$ are any two vectors perpendicular to $\alpha$. In this form the equation contains two indeterminates, and is often useful; but it is more usual to eliminate them, which may be done at once by operating by $S.\alpha$, when we obtain the equation first written.

It may also be written, by eliminating one of the indeterminates only, as

$$V\beta\rho = yV\beta\gamma = z\alpha,$$

where the form of the equation shews that $S\alpha\beta = 0$.

Similarly we see that

$$S\alpha(\rho - \beta) = 0$$

represents a plane drawn through the extremity of $\beta$ and perpendicular to $\alpha$. This, of course, may, like the last, be put into various equivalent forms.

219. The line of intersection of the two planes

$$\left.\begin{aligned} S.\alpha\ (\rho - \beta) &= 0 \\ \text{and} \quad S.\alpha_1(\rho - \beta_1) &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots \ (1)$$

contains all points whose value of $\rho$ satisfies both conditions. But we may write (§ 92), since $\alpha$, $\alpha_1$, and $V\alpha\alpha_1$ are not coplanar,

$$\rho S.\alpha\alpha_1 V\alpha\alpha_1 = V\alpha\alpha_1 S.\alpha\alpha_1\rho + V.\alpha_1 V\alpha\alpha_1 S\alpha\rho + V.V(\alpha\alpha_1)\,\alpha S\alpha_1\rho,$$

or, by the given equations,

$$-\rho T^2 V\alpha\alpha_1 = V.\alpha_1 V\alpha\alpha_1 S\alpha\beta + V.V(\alpha\alpha_1)\,\alpha S\alpha_1\beta_1 + xV\alpha\alpha_1 \ldots (2),$$

where $x$, a scalar indeterminate, is put for $S.\alpha\alpha_1\rho$ which may have any value. In practice, however, the two definite given scalar equations are generally more useful than the partially indeterminate vector-form which we have derived from them.

When both planes pass through the origin we have $\beta = \beta_1 = 0$, and obtain at once

$$\rho = xV\alpha\alpha_1$$

as the equation of the line of intersection.

220. *The plane passing through the origin, and through the line of intersection of the two planes* (1), is easily seen to have the equation

$$S\alpha_1\beta_1 S\alpha\rho - S\alpha\beta S\alpha_1\rho = 0,$$

or

$$S(\alpha S\alpha_1\beta_1 - \alpha_1 S\alpha\beta)\,\rho = 0.$$

For this is evidently the equation of a plane passing through the origin. And, if $\rho$ be such that

$$S\alpha\rho = S\alpha\beta,$$

we also have

$$S\alpha_1\rho = S\alpha_1\beta_1,$$

which are equations (1).

Hence we see that the vector

$$\alpha S\alpha_1\beta_1 - \alpha_1 S\alpha\beta$$

is perpendicular to the vector-line of intersection (2) of the two planes (1), and to every vector joining the origin with a point in that line.

The student may verify these statements as an exercise.

221. *To find the vector-perpendicular from the extremity of* $\beta$ *on the plane*

$$S\alpha\rho = 0,$$

we must note that it is necessarily parallel to $\alpha$, and hence that the value of $\rho$ for its foot is

$$\rho = \beta + x\alpha,$$

where $x\alpha$ is the vector-perpendicular in question.

Hence

$$S\alpha(\beta + x\alpha) = 0,$$

which gives $$x\alpha^2 = -S\alpha\beta,$$
or $$x\alpha = -\alpha^{-1}S\alpha\beta.$$
Similarly the vector-perpendicular from the extremity of $\beta$ on the plane
$$S\alpha(\rho-\gamma)=0$$
may easily be shewn to be
$$-\alpha^{-1}S\alpha(\beta-\gamma).$$

222. *The equation of the plane which passes through the extremities of* $\alpha$, $\beta$, $\gamma$ may be thus found. If $\rho$ be the vector of any point in it, $\rho-\alpha$, $\alpha-\beta$, and $\beta-\gamma$ lie in the plane, and therefore (§ 101)
$$S.(\rho-\alpha)(\alpha-\beta)(\beta-\gamma)=0,$$
or $$S\rho(V\alpha\beta+V\beta\gamma+V\gamma\alpha)-S.\alpha\beta\gamma=0.$$
Hence, if $$\delta=x(V\alpha\beta+V\beta\gamma+V\gamma\alpha)$$
be the vector-perpendicular from the origin on the plane containing the extremities of $\alpha$, $\beta$, $\gamma$, we have
$$\delta=(V\alpha\beta+V\beta\gamma+V\gamma\alpha)^{-1}S.\alpha\beta\gamma.$$
From this formula, whose interpretation is easy, many curious properties of a tetrahedron may be deduced by the reader. Thus, for instance, if we take the tensor of each side, and remember the result of § 100, we see that
$$T(V\alpha\beta+V\beta\gamma+V\gamma\alpha)$$
is twice the area of the base of the tetrahedron. This may be more simply proved thus. The vector area of the base is
$$\tfrac{1}{2}V(\alpha-\beta)(\gamma-\beta)=-\tfrac{1}{2}(V\alpha\beta+V\beta\gamma+V\gamma\alpha).$$

Hence the sum of the vector areas of the faces of a tetrahedron, and therefore of any solid whatever, is zero. This is the hydrostatic proposition for translational equilibrium of solids immersed in a fluid subject to no external forces.

223. Taking any two lines whose equations are
$$\rho=\beta+x\alpha,$$
$$\rho=\beta_1+x_1\alpha_1,$$
we see that $$S.\alpha\alpha_1(\rho-\delta)=0$$
is the equation of a plane parallel to both. *Which* plane, of course, depends on the value of $\delta$.

Now if $\delta=\beta$, the plane contains the first line; if $\delta=\beta_1$, the second.

Hence, if $yV\alpha\alpha_1$ be the shortest vector distance between the lines, we have

$$S.\alpha\alpha_1(\beta-\beta_1-yV\alpha\alpha_1)=0,$$

or

$$T(yV\alpha\alpha_1)=TS.(\beta-\beta_1)\,UV\alpha\alpha_1,$$

the result of § 216.

224. *Find the equation of the plane, passing through the origin, which makes equal angles with three given lines. Also find the angles in question.*

Let $\alpha$, $\beta$, $\gamma$ be unit-vectors in the directions of the lines, and let the equation of the plane be

$$S\delta\rho=0.$$

Then we have evidently

$$S\alpha\delta=S\beta\delta=S\gamma\delta=x, \text{ suppose,}$$

where

$$-\frac{x}{T\delta}$$

is the sine of each of the required angles.

But (§ 92) we have

$$\delta S.\alpha\beta\gamma=x(V\alpha\beta+V\beta\gamma+V\gamma\alpha).$$

Hence

$$S.\rho(V\alpha\beta+V\beta\gamma+V\gamma\alpha)=0$$

is the required equation; and the required sine is

$$-\frac{S.\alpha\beta\gamma}{T(V\alpha\beta+V\beta\gamma+V\gamma\alpha)}.$$

225. *Find the locus of the middle points of a series of straight lines, each parallel to a given plane and having its extremities in two fixed straight lines.*

Let

$$S\gamma\rho=0$$

be the plane, and

$$\rho=\beta+x\alpha,\quad \rho=\beta_1+x_1\alpha_1,$$

the fixed lines. Also let $x$ and $x_1$ correspond to the extremities of one of the variable lines, $\varpi$ being the vector of its middle point.

Then, obviously,

$$2\varpi=\beta+x\alpha+\beta_1+x_1\alpha_1.$$

Also

$$S\gamma(\beta-\beta_1+x\alpha-x_1\alpha_1)=0.$$

This gives a linear relation between $x$ and $x_1$, so that, if we substitute for $x_1$ in the preceding equation, we obtain a result of the form

$$\varpi=\delta+x\epsilon,$$

where $\delta$ and $\epsilon$ are known vectors. The required locus is, therefore, a straight line.

226. *Three planes meet in a point, and through the line of intersection of each pair a plane is drawn perpendicular to the third; prove that these planes pass through the same line.*

Let the point be taken as origin, and let the equations of the planes be

$$S\alpha\rho = 0, \quad S\beta\rho = 0, \quad S\gamma\rho = 0.$$

The line of intersection of the first two is $\parallel V\alpha\beta$, and therefore the normal to the first of the new planes is

$$V.\gamma V\alpha\beta.$$

Hence the equation of this plane is

$$S.\rho V.\gamma V\alpha\beta = 0,$$

or

$$S\beta\rho S\alpha\gamma - S\alpha\rho S\beta\gamma = 0,$$

and those of the other two planes may be easily formed from this by cyclical permutation of $\alpha, \beta, \gamma$.

We see at once that any two of these equations give the third by addition or subtraction, which is the proof of the theorem.

227. *Given any number of points $A$, $B$, $C$, &c., whose vectors (from the origin) are $\alpha_1$, $\alpha_2$, $\alpha_3$, &c., find the plane through the origin for which the sum of the squares of the perpendiculars let fall upon it from these points is a maximum or minimum.*

Let

$$S\varpi\rho = 0$$

be the required equation, with the condition (evidently allowable)

$$T\varpi = 1.$$

The perpendiculars are (§ 221) $-\varpi^{-1}S\varpi\alpha_1$, &c.

Hence

$$\Sigma S^2\varpi\alpha$$

is a maximum. This gives

$$\Sigma.S\varpi\alpha S\alpha d\varpi = 0;$$

and the condition that $\varpi$ is a unit-vector gives

$$S\varpi d\varpi = 0.$$

Hence, as $d\varpi$ may have any of an infinite number of values, these equations cannot be consistent unless

$$\Sigma.\alpha S\alpha\varpi = x\varpi,$$

where $x$ is a scalar.

The values of $\alpha$ are known, so that if we put

$$\Sigma . \alpha S\alpha\varpi = \phi\varpi,$$

$\phi$ is a given self-conjugate linear and vector function, and therefore $x$ has three values ($g_1$, $g_2$, $g_3$, § 175) which correspond to three mutually perpendicular values of $\varpi$. For one of these there is a maximum, for another a minimum, for the third a maximum-minimum, in the most general case when $g_1$, $g_2$, $g_3$ are all different.

228. The following beautiful problem is due to Maccullagh. *Of a system of three rectangular vectors, passing through the origin, two lie on given planes, find the locus of the third.*

Let the rectangular vectors be $\varpi$, $\rho$, $\sigma$. Then by the conditions of the problem

$$S\varpi\rho = S\rho\sigma = S\sigma\varpi = 0,$$

and $$S\alpha\varpi = 0, \qquad S\beta\rho = 0.$$

The solution depends on the elimination of $\rho$ and $\varpi$ among these five equations. [This would, in general, be impossible, as $\rho$ and $\varpi$ between them involve *six* unknown scalars; but, as the tensors are (by the very form of the equations) not involved, the five given equations are necessary and sufficient to eliminate the four unknown scalars which are really involved. Formally to complete the requisite number of equations we might write

$$T\varpi = a, \quad T\rho = b,$$

but $a$ and $b$ may have any values whatever.]

From $$S\alpha\varpi = 0, \quad S\sigma\varpi = 0,$$

we have $$\varpi = xV\alpha\sigma.$$

Similarly, from $$S\beta\rho = 0, \quad S\sigma\rho = 0,$$

we have $$\rho = yV\beta\sigma.$$

Substitute in the remaining equation

$$S\varpi\rho = 0,$$

and we have $$S . V\alpha\sigma V\beta\sigma = 0,$$

or $$S\alpha\sigma S\beta\sigma - \sigma^2 S\alpha\beta = 0,$$

the required equation. As will be seen in next Chapter, this is a cone of the second degree whose circular sections are perpendicular to $\alpha$ and $\beta$. [The disappearance of $x$ and $y$ in the elimination instructively illustrates the note above.]

## EXAMPLES TO CHAPTER VII.

1. What propositions of Euclid are proved by the mere *form* of the equation

$$\rho = (1 - x)\,\alpha + x\beta,$$

which denotes the line joining any two points in space?

2. Shew that the chord of contact, of tangents to a parabola which meet at right angles, passes through a fixed point.

3. Prove the chief properties of the circle (as in *Euclid*, III.) from the equation

$$\rho = \alpha \cos\theta + \beta \sin\theta;$$

where $T\alpha = T\beta$, and $S\alpha\beta = 0$.

4. What locus is represented by the equation

$$S^2\alpha\rho + \rho^2 = 0,$$

where $T\alpha = 1$?

5. What is the condition that the lines

$$V\alpha\rho = \beta, \quad V\alpha_1\rho = \beta_1,$$

intersect? If this is not satisfied, what is the shortest distance between them?

6. Find the equation of the plane which contains the two parallel lines

$$V\alpha(\rho - \beta) = 0, \quad V\alpha(\rho - \beta_1) = 0.$$

7. Find the equation of the plane which contains

$$V\alpha(\rho - \beta) = 0,$$

and is perpendicular to $S\gamma\rho = 0$.

8. Find the equation of a straight line passing through a given point, and making a given angle with a given plane.

Hence form the general equation of a right cone.

9. What conditions must be satisfied with regard to a number of given lines in space that it may be possible to draw through each of them a plane in such a way that these planes may intersect in a common line?

10. Find the equation of the locus of a point the sum of the squares of whose distances from a number of given planes is constant.

11. Substitute "lines" for "planes" in (10).

12. Find the equation of the plane which bisects, at right angles, the shortest distance between two given lines.

Find the locus of a point in this plane which is equidistant from the given lines.

13. Find the conditions that the simultaneous equations

$$S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c,$$

may represent a line, and not a point.

14. What is represented by the equations

$$(S\alpha\rho)^2 = (S\beta\rho)^2 = (S\gamma\rho)^2,$$

where $\alpha$, $\beta$, $\gamma$ are any three vectors?

15. Find the equation of the plane which passes through two given points and makes a given angle with a given plane.

16. Find the area of the triangle whose corners have the vectors $\alpha$, $\beta$, $\gamma$.

Hence form the equation of a circular cylinder whose axis and radius are given.

17. (Hamilton, *Bishop Law's Premium Ex.*, 1858.)

(*a*) Assign some of the transformations of the expression

$$\frac{V\alpha\beta}{\beta - \alpha},$$

where $\alpha$ and $\beta$ are the vectors of two given points $A$ and $B$.

(*b*) The expression represents the vector $\gamma$, or $\overline{OC}$, of a point $C$ in the straight line $AB$.

(*c*) Assign the position of this point $C$.

18. (*Ibid.*)

(*a*) If $\alpha$, $\beta$, $\gamma$, $\delta$ be the vectors of four points, $A$, $B$, $C$, $D$, what is the condition for those points being in one plane?

(*b*) When these four vectors from one origin do not thus terminate upon one plane, what is the expression for the volume of the pyramid, of which the four points are the corners?

(*c*) Express the perpendicular $\delta$ let fall from the origin $O$ on the plane $ABC$, in terms of $\alpha$, $\beta$, $\gamma$.

19. Find the locus of a point equidistant from the three planes

$$S\alpha\rho = 0, \quad S\beta\rho = 0, \quad S\gamma\rho = 0.$$

20. If three mutually perpendicular vectors be drawn from a point to a plane, the sum of the reciprocals of the squares of their lengths is independent of their directions.

21. Find the general form of the equation of a plane from the condition (which is to be assumed as a definition) that any two planes intersect in a single straight line.

22. Prove that the sum of the vector areas of the faces of any polyhedron is zero.

# CHAPTER VIII.

## THE SPHERE AND CYCLIC CONE.

229. AFTER that of the plane the equations next in order of simplicity are those of the sphere, and of the cone of the second order. To these we devote a short Chapter as a valuable preparation for the study of surfaces of the second order in general.

230. The equation

$$T\rho = T\alpha,$$

or

$$\rho^2 = \alpha^2,$$

denotes that the length of $\rho$ is the same as that of a given vector $\alpha$, and therefore belongs to a sphere of radius $T\alpha$ whose centre is the origin. In § 107 several transformations of this equation were obtained, some of which we will repeat here with their interpretations.

Thus

$$S(\rho + \alpha)(\rho - \alpha) = 0$$

shews that the chords drawn from any point on the sphere to the extremities of a diameter (whose vectors are $\alpha$ and $-\alpha$) are at right angles to each other.

$$T(\rho + \alpha)(\rho - \alpha) = 2TV\alpha\rho$$

shews that the rectangle under these chords is four times the area of the triangle two of whose sides are $\alpha$ and $\rho$.

$$\rho = (\rho + \alpha)^{-1}\alpha(\rho + \alpha) \text{ (see § 105)}$$

shews that the angle at the centre in any circle is double that at the circumference standing on the same arc. All these are easy consequences of the processes already explained for the interpretation of quaternion expressions.

231. If the centre of a sphere be at the extremity of $\alpha$, the equation may be written

$$T(\rho - \alpha) = T\beta,$$

which is the most general form.

If $$T\alpha = T\beta,$$

or $$\alpha^2 = \beta^2,$$

in which case the origin is a point on the surface of the sphere, this becomes

$$\rho^2 - 2S\alpha\rho = 0.$$

From this, in the form

$$S\rho(\rho - 2\alpha) = 0$$

another proof that the angle in a semicircle is a right angle is derived at once.

232. The converse problem is—*Find the locus of the feet of perpendiculars let fall from a given point*, $\rho = \beta$, *on planes passing through the origin.*

Let $$S\alpha\rho = 0$$

be one of the planes, then (§ 221) the vector-perpendicular is

$$-\alpha^{-1}S\alpha\beta,$$

and, for the locus of its foot,

$$\rho = \beta - \alpha^{-1}S\alpha\beta,$$
$$= \alpha^{-1}V\alpha\beta.$$

[This is an example of a peculiar form in which quaternions sometimes give us the equation of a surface. The equation is a vector one, or equivalent to three scalar equations; but it involves the undetermined vector $\alpha$ in such a way as to be equivalent to only two indeterminates (as the tensor of $\alpha$ is evidently not involved). To put the equation in a more immediately interpretable form, $\alpha$ must be eliminated, and the remarks just made shew this to be possible.]

Now $$(\rho - \beta)^2 = \alpha^{-2}S^2\alpha\beta,$$

and (operating by $S.\beta$ on the value of $\rho$ above)

$$S\beta\rho - \beta^2 = -\alpha^{-2}S^2\alpha\beta.$$

Adding these equations, we get

$$\rho^2 - S\beta\rho = 0,$$

or $$T\left(\rho - \frac{\beta}{2}\right) = T\frac{\beta}{2},$$

so that, as is evident, the locus is the sphere of which $\beta$ is a diameter.

233. *To find the intersection of the two spheres*

$$T(\rho-\alpha)=T\beta,$$

and
$$T(\rho-\alpha_1)=T\beta_1,$$

square the equations, and subtract, and we have

$$2S(\alpha-\alpha_1)\rho=\alpha^2-\alpha_1^2-(\beta^2-\beta_1^2),$$

which is the equation of a plane, perpendicular to $\alpha-\alpha_1$, the vector joining the centres of the spheres. This is always a real plane whether the spheres intersect or not. It is, in fact, what is called their *Radical Plane.*

234. *Find the locus of a point the ratio of whose distances from two given points is constant.*

Let the given points be $O$ and $A$, the extremities of the vector $\alpha$. Also let $P$ be the required point in any of its positions, and $\overline{OP}=\rho$.

Then, at once, if $n$ be the ratio of the lengths of the two lines,

$$T(\rho-\alpha)=nT\rho.$$

This gives
$$\rho^2-2S\alpha\rho+\alpha^2=n^2\rho^2,$$

or, by an easy transformation,

$$T\left(\rho-\frac{\alpha}{1-n^2}\right)=T\left(\frac{n\alpha}{1-n^2}\right).$$

Thus the locus is a sphere whose radius is $T\left(\frac{n\alpha}{1-n^2}\right)$, and whose centre is at $B$, where $\overline{OB}=\frac{\alpha}{1-n^2}$, a definite point in the line $OA$.

235. *If in any line, $OP$, drawn from the origin to a given plane, $OQ$ be taken such that $OQ \,.\, OP$ is constant, find the locus of $Q$.*

Let
$$S\alpha\rho=a^2$$

be the equation of the plane, $\varpi$ a vector of the required surface. Then, by the conditions,

$$T\varpi T\rho=\text{constant}=b^2 \text{ (suppose)},$$

and
$$U\varpi=U\rho.$$

From these
$$\rho=\frac{b^2U\varpi}{T\varpi}=-\frac{b^2\varpi}{\varpi^2}.$$

Substituting in the equation of the plane, we have

$$a^2\varpi^2+b^2S\alpha\varpi=0,$$

which shews that the locus is a sphere, the origin being situated on it at the point farthest from the given plane.

236. *Find the locus of points the sum of the squares of whose distances from a set of given points is a constant quantity. Find also the least value of this constant, and the corresponding locus.*

Let the vectors from the origin to the given points be $\alpha_1$, $\alpha_2$, ......$\alpha_n$, and to the sought point $\rho$, then

$$-c^2 = (\rho - \alpha_1)^2 + (\rho - \alpha_2)^2 + \ldots\ldots + (\rho - \alpha_n)^2,$$
$$= n\rho^2 - 2S\rho\Sigma\alpha + \Sigma(\alpha^2).$$

Otherwise $$\left(\rho - \frac{\Sigma\alpha}{n}\right)^2 = -\frac{c^2 + \Sigma(\alpha^2)}{n} + \frac{(\Sigma\alpha)^2}{n^2},$$

the equation of a sphere the vector of whose centre is $\frac{\Sigma\alpha}{n}$, i.e. whose centre is the mean of the system of given points.

Suppose the origin to be placed at the mean point, the equation becomes

$$\rho^2 = -\frac{c^2 + \Sigma(\alpha^2)}{n} \text{ (because } \Sigma\alpha = 0, \text{ § 31 } (e)).$$

The right-hand side is negative, and therefore the equation denotes a real surface, if

$$c^2 > \Sigma T\alpha^2,$$

as might have been expected. When these quantities are equal, the locus becomes a point, viz. the new origin, or the mean point of the system.

237. If we differentiate the equation

$$T\rho = T\alpha$$

we get $$S\rho d\rho = 0.$$

Hence (§ 144), $\rho$ is *normal* to the surface at its extremity, a well-known property of the sphere.

If $\varpi$ be any point in the plane which touches the sphere at the extremity of $\rho$, $\varpi - \rho$ is a line in the tangent plane, and therefore perpendicular to $\rho$. So that

$$S\rho(\varpi - \rho) = 0,$$

or $$S\varpi\rho = -T\rho^2 = \alpha^2$$

is the equation of the tangent plane.

238. If this plane pass through a given point $B$, whose vector is $\beta$, we have

$$S\beta\rho = \alpha^2.$$

This is the equation of a plane, perpendicular to $\beta$, and cutting from it a portion whose length is

$$\frac{T\alpha^2}{T\beta}.$$

If this plane pass through a fixed point whose vector is $\gamma$ we must have

$$S\beta\gamma = \alpha^2,$$

so that the locus of $\beta$ is a plane. These results contain all the ordinary properties of poles and polars with regard to a sphere.

239. A line drawn parallel to $\gamma$, from the extremity of $\beta$, has the equation

$$\rho = \beta + x\gamma.$$

This meets the sphere

$$\rho^2 = \alpha^2$$

in points for which $x$ has the values given by the equation

$$\beta^2 + 2xS\beta\gamma + x^2\gamma^2 = \alpha^2.$$

The values of $x$ are imaginary, that is, there is no intersection, if

$$\alpha^2\gamma^2 + V^2\beta\gamma < 0.$$

The values are equal, or the line touches the sphere, if

$$\alpha^2\gamma^2 + V^2\beta\gamma = 0,$$

or

$$S^2\beta\gamma = \gamma^2(\beta^2 - \alpha^2).$$

This is the equation of a cone similar and similarly situated to the cone of tangent-lines drawn to the sphere, but its vertex is at the centre. That the equation represents a cone is obvious from the fact that it is *homogeneous* in $T\gamma$, i.e. that it is independent of the length of the vector $\gamma$.

[It may be remarked that from the form of the above equation we see that, if $x$ and $x'$ be its roots, we have

$$(xT\gamma)(x'T\gamma) = \alpha^2 - \beta^2,$$

which is *Euclid*, III. 35, 36, extended to a sphere.]

240. *Find the locus of the foot of the perpendicular let fall from a given point of a sphere on any tangent-plane.*

Taking the centre as origin, the equation of any tangent-plane may be written

$$S\varpi\rho = \alpha^2.$$

The perpendicular must be parallel to $\rho$, so that, if we suppose it drawn from the extremity of $\alpha$ (which is a point on the sphere) we have as one value of $\varpi$

$$\varpi = \alpha + x\rho.$$

From these equations, with the help of that of the sphere

$$\rho^2 = \alpha^2,$$

we must eliminate $\rho$ and $x$.

We have by operating on the vector equation by $S.\varpi$

$$\varpi^2 = S\alpha\varpi + xS\varpi\rho$$
$$= S\alpha\varpi + x\alpha^2.$$

Hence $$\rho = \frac{\varpi - \alpha}{x} = \frac{\alpha^2(\varpi - \alpha)}{\varpi^2 - S\alpha\varpi}.$$

Taking the tensors, we have

$$(\varpi^2 - S\alpha\varpi)^2 = \alpha^2(\varpi - \alpha)^2,$$

the required equation. It may be put in the form

$$S^2\varpi U(\varpi - \alpha) = -\alpha^2,$$

and the interpretation of this (viz. that the projection of $\varpi$ on $(\varpi - \alpha)$ is of constant length) gives at once a characteristic property of the surface formed by the rotation of the *Cardioid* about its axis of symmetry.

If the perpendiculars be let fall from a point, $\beta$, not on the sphere, it is easy to see that the equation of the locus is

$$S^2\varpi U(\varpi - \beta) = -\alpha^2,$$

whose interpretation is equally easy.

241. We have seen that a sphere, referred to any point whatever as origin, has the equation

$$T(\rho - \alpha) = T\beta.$$

Hence, *to find the rectangle under the segments of a chord drawn through any point,* we may put

$$\rho = x\gamma;$$

where $\gamma$ is *any* unit-vector whatever. This gives

$$x^2\gamma^2 - 2xS\alpha\gamma + \alpha^2 = \beta^2,$$

and the product of the two values of $x$ is

$$-\frac{\beta^2 - \alpha^2}{\gamma^2} = -\alpha^2 + \beta^2.$$

This is positive, or the vector-chords are drawn in the *same* direction, if

$$T\beta < T\alpha,$$

i.e. if the origin is outside the sphere.

242. *A, B are fixed points; and, O being the origin and P a point in space,*

$$AP^2 + BP^2 = OP^2;$$

*find the locus of P, and explain the result when ∠ AOB is a right, or an obtuse, angle.*

Let $\overline{OA} = \alpha$, $\overline{OB} = \beta$, $\overline{OP} = \rho$, then

$$(\rho - \alpha)^2 + (\rho - \beta)^2 = \rho^2,$$

or
$$\rho^2 - 2S(\alpha + \beta)\rho = -(\alpha^2 + \beta^2),$$

or
$$T\{\rho - (\alpha + \beta)\} = \sqrt{(-2S\alpha\beta)}.$$

While $S\alpha\beta$ is negative, that is, while $\angle AOB$ is acute, the locus is a sphere whose centre has the vector $\alpha + \beta$. If $S\alpha\beta = 0$, or $\angle AOB = \pi/2$, the locus is reduced to the point

$$\rho = \alpha + \beta.$$

If $\angle AOB > \pi/2$ there is no point which satisfies the conditions.

243. *Describe a sphere, with its centre in a given line, so as to pass through a given point and touch a given plane.*

Let $x\alpha$, where $x$ is an undetermined scalar, be the vector of the centre, $r$ the radius of the sphere, $\beta$ the vector of the given point, and

$$S\gamma\rho = a$$

the equation of the given plane.

The vector-perpendicular from the point $x\alpha$ on the given plane is (§ 221)

$$(a - xS\gamma\alpha)\gamma^{-1}.$$

Hence, to determine $x$ and $r$ we have the equations

$$T.(a - xS\gamma\alpha)\gamma^{-1} = T(x\alpha - \beta) = r,$$

so that there are, in general, two solutions. It will be a good exercise for the student to find from these equations the condition that there may be no solution, or two coincident ones.

244. *Describe a sphere whose centre is in a given line, and which passes through two given points.*

Let the vector of the centre be $x\alpha$, as in last section, and let the vectors of the points be $\beta$ and $\gamma$. Then, at once,

$$T(\gamma - x\alpha) = T(\beta - x\alpha) = r.$$

Here there is but *one* sphere, except in the particular case when we have

$$T\gamma = T\beta, \quad \text{and} \quad S\alpha\gamma = S\alpha\beta,$$

in which case there is an infinite number.

The student should carefully compare the results of this section with those of the last, so as to discover why in general two solutions are indicated as possible in the one problem, and only one in the other.

245. *A sphere touches each of two straight lines, which do not meet: find the locus of its centre.*

We may take the origin at the middle point of the shortest distance (§ 216) between the given lines, and their equations will then be

$$\rho = \alpha + x\beta,$$

$$\rho = -\alpha + x_1\beta_1,$$

where we have, of course,

$$S\alpha\beta = 0, \quad S\alpha\beta_1 = 0.$$

Let $\sigma$ be the vector of the centre, $\rho$ that of any point, of one of the spheres, and $r$ its radius; its equation is

$$T(\rho - \sigma) = r.$$

Since the two given lines are tangents, the following equations in $x$ and $x_1$ must have pairs of equal roots,

$$T(\alpha + x\beta - \sigma) = r,$$

$$T(-\alpha + x_1\beta_1 - \sigma) = r.$$

The equality of the roots in each gives us the conditions

$$S^2\beta\sigma = \beta^2\{(\alpha - \sigma)^2 + r^2\},$$

$$S^2\beta_1\sigma = \beta_1^2\{(\alpha + \sigma)^2 + r^2\}.$$

Eliminating $r$ we obtain

$$\beta^{-2}S^2\beta\sigma - \beta_1^{-2}S^2\beta_1\sigma = (\alpha - \sigma)^2 - (\alpha + \sigma)^2 = -4S\alpha\sigma,$$

which is the equation of the required locus.

[As we have not, so far, entered on the consideration of the quaternion form of the equations of the various surfaces of the second order, we may translate this into Cartesian cöordinates to find its meaning. If we take cöordinate axes of $x, y, z$ respectively parallel to $\beta, \beta_1, \alpha$, it becomes at once

$$(x + my)^2 - (y + mx)^2 = pz,$$

where $m$ and $p$ are constants; and shews that the locus is a hyperbolic paraboloid. Such transformations, which are exceedingly simple in all cases, will be of frequent use to the student who is proficient in Cartesian geometry, in the early stages of his study of quaternions. As he acquires a practical knowledge of the new calculus, the need of such assistance will gradually cease to be felt.]

Simple as the above solution is, quaternions enable us to give one vastly simpler. For the problem may be thus stated—*Find the locus of the point whose distances from two given lines are equal.* And, with the above notation, the equality of the perpendiculars is expressed (§ 214) by

$$TV.(\alpha-\sigma)\,U\beta = TV.(\alpha+\sigma)\,U\beta_1,$$

which is easily seen to be equivalent to the equation obtained above.

246. *Two spheres being given, shew that spheres which cut them at given angles cut at right angles another fixed sphere.*

If $c$ be the distance between the centres of two spheres whose radii are $a$ and $b$, the cosine of the angle of intersection is evidently

$$\frac{a^2+b^2-c^2}{2ab}.$$

Hence, if $\alpha$, $\alpha_1$, and $\rho$ be the vectors of the centres, and $a$, $a_1$, $r$ the radii, of the two fixed, and of one of the variable, spheres; $A$ and $A_1$ the angles of intersection, we have

$$(\rho-\alpha)^2+a^2+r^2=2ar\cos A,$$
$$(\rho-\alpha_1)^2+a_1{}^2+r^2=2a_1r\cos A_1.$$

Eliminating the first power of $r$, we evidently must obtain a result such as

$$(\rho-\epsilon)^2+e^2+r^2=0,$$

where (by what precedes) $\epsilon$ is the vector of the centre, and $e$ the radius, of a fixed sphere

$$(\rho-\epsilon)^2+e^2=0,$$

which is cut at right angles by all the varying spheres. By effecting the elimination exactly we easily find $e$ and $\epsilon$ in terms of given quantities.

247. If two vectors divide one another into parts $\alpha$ and $-e\alpha$, $\rho$ and $-\rho$, respectively, the lines joining the free ends meet in

$$\varpi=\frac{(1+e)\,\rho-2e\alpha}{1-e}.$$

We may write this as

$$e_1\rho=\varpi+\alpha_1,$$

if
$$e_1=\frac{1+e}{1-e},\qquad \alpha_1=\frac{2e\alpha}{1-e}.$$

For another such pair of vectors, passing through the same point, we have, say,

$$f_1\sigma = \varpi + \beta_1.$$

Thus the plane quadrilateral, whose diagonals are made up respectively of the complete $\alpha$ and $\beta$ vectors, will be projected (by lines from the point $\varpi$) as a square on the plane of $\rho$, $\sigma$, provided

$$T\rho = T\sigma, \text{ and } S\rho\sigma = 0.$$

That is, provided

$$f_1^2 (\varpi + \alpha_1)^2 = e_1^2 (\varpi + \beta_1)^2 \dotfill (1),$$

$$S.(\varpi + \alpha_1)(\varpi + \beta_1) = 0 \dotfill (2).$$

These equations obviously belong to spheres which intersect one another at right angles. For the centres are at

$$-\frac{1}{f_1^2 - e_1^2}(f_1^2\alpha_1 - e_1^2\beta_1) \text{ and } -\tfrac{1}{2}(\alpha_1 + \beta_1).$$

Thus the distance between the centres is

$$T.\frac{e_1^2 + f_1^2}{2(f_1^2 - e_1^2)}(\alpha_1 - \beta_1);$$

and this is obviously less than the sum, and greater than the difference, of the radii

$$T.\frac{e_1 f_1}{f_1^2 - e_1^2}(\alpha_1 - \beta_1), \text{ and } T.\tfrac{1}{2}(\alpha_1 - \beta_1).$$

And because its square is equal to the sum of their squares, the spheres intersect at right angles. Hence

*The locus of the points, from which a plane (uncrossed) quadrilateral can be projected as a square, is a circle whose centre is in the plane of the quadrilateral, and whose plane is perpendicular to that plane.*

To find the points (if any) of this circle from which the quadrilateral *is seen* as a square, we must introduce the additional conditions

$$S\varpi\rho = 0, \quad S\varpi\sigma = 0,$$

or

$$S\varpi(\varpi + \alpha_1) = 0, \quad S\varpi(\varpi + \beta_1) = 0 \dotfill (3).$$

Hence the points lie on each of two spheres which pass through the origin:—i.e. the intersection of the diagonals.

[If we eliminate $\varpi$ among the four equations (1), (2), (3), we find the condition

$$S(\alpha_1 - \beta_1)(f_1^2\alpha_1 + e_1^2\beta_1) = 0.$$

This we leave to the student.]

Another mode of solving the last problem, viz. *to find the points from which a given plane quadrilateral is seen as a square*, consists in expressing that the four portions of the diagonals subtend equal angles, and that the planes containing them are at right angles to one another.

The first condition gives, with the notation of the beginning of this section,

$$S.\varpi U(\varpi-\alpha)=S.\varpi U(\varpi+e\alpha)$$
$$=S.\varpi U(\varpi-\beta)=S.\varpi U(\varpi+f\beta)\ldots\ldots(4).$$

The second condition is

$$S.\,V\varpi\alpha V\varpi\beta=0,$$

or

$$\varpi^2 S\alpha\beta - S\alpha\varpi S\beta\varpi=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

the cone whose cyclic normals are $\alpha$, $\beta$.

[It will be excellent practice for the student to shew that (4) and (5) are equivalent to (1), (2), (3). Thus, in particular, the first equality in (4)

$$S.\varpi U(\varpi-\alpha)=S.\varpi U(\varpi+e\alpha),$$

is equivalent to the first of (3), viz.

$$S.\varpi(\varpi+\alpha_1)=0.]$$

It is obvious that the solution of the first problem in this section gives at once the means of solving the problem of *projecting an ellipse into a circle, so that any given (internal) point may be projected as the centre of the circle.* And numerous other consequences follow, which may be left to the reader.

248. *To inscribe in a given sphere a closed polygon, plane or gauche, whose sides shall be parallel respectively to each of a series of given vectors.*

Let

$$T\rho=1$$

be the sphere, $\alpha, \beta, \gamma, \ldots\ldots, \eta, \theta$ the vectors, $n$ in number, and let $\rho_1, \rho_2, \ldots\ldots \rho_n$, be the vector-radii drawn to the angles of the polygon.

Then

$$\rho_2-\rho_1=x_1\alpha, \text{ \&c., \&c.}$$

From this, by operating by $S.(\rho_2+\rho_1)$, we get

$$\rho_2^{\,2}-\rho_1^{\,2}=0=S\alpha\rho_2+S\alpha\rho_1.$$

Also

$$0=V\alpha\rho_2-V\alpha\rho_1.$$

Adding, we get

$$0=\alpha\rho_2+K\alpha\rho_1=\alpha\rho_2+\rho_1\alpha.$$

Hence

$$\rho_2=-\alpha^{-1}\rho_1\alpha,$$

or, if we please,

$$\rho_2=-\alpha\rho_1\alpha^{-1}.$$

[This might have been written down at once from the result of § 105.]

Similarly $\rho_3 = -\beta^{-1}\rho_2\beta = \beta^{-1}\alpha^{-1}\rho_1\alpha\beta$, &c.

Thus, finally, since the polygon is closed,

$$\rho_{n+1} = \rho_1 = (-)^n\,\theta^{-1}\eta^{-1}\ldots\ldots\beta^{-1}\alpha^{-1}\rho_1\alpha\beta\ldots\ldots\eta\theta.$$

We may suppose the tensors of $\alpha$, $\beta\ldots\ldots\eta$, $\theta$ to be each unity. Hence, if

$$a = \alpha\beta\ldots\ldots\eta\theta,$$

we have
$$a^{-1} = \theta^{-1}\eta^{-1}\ldots\ldots\beta^{-1}\alpha^{-1},$$

which is a known quaternion; and thus our condition becomes

$$\rho_1 = (-)^n\,a^{-1}\rho_1 a.$$

This divides itself into two cases, according as $n$ is an even or an odd number.

If $n$ be even, we have

$$a\rho_1 = \rho_1 a.$$

Removing the common part $\rho_1 Sa$, we have

$$V\rho_1 Va = 0.$$

This gives one determinate direction, $\pm Va$, for $\rho_1$; and shews that there are two, and only two, solutions.

If $n$ be odd, we have

$$a\rho_1 = -\rho_1 a,$$

which (operating, for instance, by $S.\rho_1$) requires that we have

$$Sa = 0,$$

i.e. that there may be a solution, $a$ must be a vector.

Hence
$$Sa\rho_1 = 0,$$

and therefore $\rho_1$ may be drawn to any point in the great circle of the unit-sphere whose pole is on the vector $a$.

249. To illustrate these results, let us take first the case of $n = 3$. We must have

$$S.\alpha\beta\gamma = 0,$$

or the three given vectors must (as is obvious on other grounds) be parallel to one plane. Here $\alpha\beta\gamma$, which lies in this plane, is (§ 106) the vector-tangent at the first corner of each of the inscribed triangles; and is obviously perpendicular to the vector drawn from the centre to that corner.

If $n = 4$, we have

$$\rho_1 \parallel V.\alpha\beta\gamma\delta,$$

as might have been at once seen from § 106.

250. Hamilton has given (*Lectures*, p. 674 and *Appendix* C.), an ingenious process by which the above investigation is rendered applicable to the more difficult problem in which each side of the inscribed polygon is to pass through a given point instead of being parallel to a given line. His process, which (see his *Life*, Vol. III., pp. 88, 426) he evidently considered as a specially tough piece of analysis, depends upon the integration of a linear equation in finite differences.

The gist of Hamilton's method is (briefly) as follows (*Lectures*, § 676):

Let the (unit) vectors to the corners of the polygon be, as above, $\rho_1, \rho_2, \ldots\ldots \rho_n$. Also let $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ be the points through which the successive sides are to pass. The sides are respectively parallel to the vectors

$$\alpha_1 - \rho_1, \quad \alpha_2 - \rho_2, \ldots\ldots, \alpha_n - \rho_n,$$

which correspond to $\alpha, \beta, \ldots\ldots \theta$, of § 248. Hence, if we write

$$q_1 = \alpha_1 - \rho_1,$$
$$q_2 = (\alpha_2 - \rho_2)\, q_1,$$
$$q_3 = (\alpha_3 - \rho_3)\, q_2, \text{ \&c.}$$

we have (as in that section), since the expressions are independent of the tensors of the $q$s,

$$\rho_2 = -\, q_1 \rho_1 q_1^{-1}$$
$$\rho_3 = +\, q_2 \rho_1 q_2^{-1}$$
$$\rho_4 = -\, q_3 \rho_1 q_3^{-1}, \text{ \&c.}$$

These give, generally, (with the condition $\rho_1^2 = -1$)

$$q_m = r_m + (-1)^m s_m \rho_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where
$$r_m = \alpha_m r_{m-1} + s_{m-1},$$
$$s_m = r_{m-1} - \alpha_m s_{m-1}.$$

[We may easily eliminate $s$, by the use of the separable symbol $D$ or $1 + \Delta$, but this leads to a troublesome species of equation of second differences. Hamilton ingeniously avoids this by the use of biquaternions.]

Putting $\iota$ for the algebraic $\sqrt{-1}$, we have

$$r_m + \iota s_m = (\alpha_m + \iota)(r_{m-1} - \iota s_{m-1}),$$

(where, as usual, we have a second equation by changing throughout the sign of $\iota$).

The complete solution of this equation is, of course, obtained at once in the form of a finite product. But it is sufficient to know some of its characteristic properties only.

The squared tensor is

$$T^2r_m - T^2s_m + 2\iota S.r_mKs_m = (T^2\alpha_m - 1)(T^2r_{m-1} - T^2s_{m-1} - 2\iota S.r_{m-1}Ks_{m-1}),$$

so that, by equating real and imaginary parts, we have

$$T^2r_m - T^2s_m = (T^2\alpha_m - 1)\,(T^2r_{m-1} - T^2s_{m-1}),$$

$$S.r_mKs_m = -(T^2\alpha_m - 1)\,S.r_{m-1}Ks_{m-1}.$$

But, by the value of $q_1$ above, we have $r_1 = \alpha_1$, $s_1 = 1$, so that

$$T^2r_m - T^2s_m = (T^2\alpha_m - 1)\,(T^2\alpha_{m-1} - 1)\ldots\ldots(T^2\alpha_1 - 1),$$

$$S.r_mKs_m = 0.$$

Thus it appears that we may write

$$q_n = b + \beta + (-1)^n\,(c + \gamma)\,\rho_1 \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

with the condition

$$bc = S\beta\gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

But, if we write, putting $\iota$ instead of $\rho_1$ in $q_n$,

$$Q = b + \beta + (-1)^n\,(c + \gamma)\,\iota,$$

we have

$$\frac{V}{S}Q = \lambda + \mu\iota, \text{ suppose,}$$

where $\lambda$ and $\mu$ are real vectors whose values can be calculated from the data. And we now have

$$\rho_{n+1} = \rho_1 = (-1)^n\,(1 + \lambda + \mu\rho_1)\,\rho_1\,(1 + \lambda + \mu\rho_1)^{-1}$$

$$= (-1)^n\,\frac{(1 + \lambda)\,\rho_1 - \mu}{1 + \lambda + \mu\rho_1}.$$

When $n$ is odd, this gives at once

$$(1 + S\mu\rho_1)\,\rho_1 + S\lambda\rho_1 = 0\,;$$

which, since $\rho_1$ does not vanish, leads to the two equations

$$S\lambda\rho_1 = 0,$$

$$S\mu\rho_1 = -1.$$

These planes intersect in a line, whose intersections with the unit-sphere give the possible extremities of the required first radius.

When $n$ is even, we have

$$V\lambda\rho_1 = \mu + \rho_1S\mu\rho_1 = V.\rho_1V\mu\rho_1,$$

or

$$V.\rho_1\,(\lambda + V\mu\rho_1) = 0. \quad (\S\ 199.)$$

With the notation of (2) the condition (3) becomes

$$(-1)^n\,\frac{bc}{b^2 - c^2} + \frac{S\lambda\mu}{\lambda^2 - \mu^2 - 1} = 0.$$

For further details, see especially *Appendices* B and C to the *Lectures.*

By an immediate application of the linear and vector function of Chapter V., the above solutions may be at once extended to any central surface of the second order.

250*. The quaternions which Hamilton employed (as above) were such as change the radius to one corner of the polygon into that to the next by a *conical* rotation. It may be interesting and useful to the student to compare with Hamilton's solution the following, which employs the quaternions which *directly* turn one side of the polygon to lie along the next. The successive sides are expressed as ratios of one of these quaternions to the next.

Let $\rho_1$, $\rho_2$, &c., $\rho_n$ be (unit) vectors drawn from the centre of the sphere to the corners of the polygon; $\alpha_1, \alpha_2, \ldots \alpha_n$, the points through which the successive sides are to pass. Then (by Euclid) we have

$$(\rho_2-\alpha_1)(\rho_1-\alpha_1)=1+\alpha_1^2=A_1, \text{ suppose.}$$

$$(\rho_3-\alpha_2)(\rho_2-\alpha_2)=1+\alpha_2^2=A_2,$$

$$\&c. \qquad = \qquad \&c.$$

$$(\rho_{n+1}-\alpha_n)(\rho_n-\alpha_n)=1+\alpha_n^2=A_n.$$

These equations ensure that if the tensor of any one of the $\rho$s be unit, those of all the others shall also be units. Thus we have merely to eliminate $\rho_2, \ldots, \rho_n$; and then remark that (for the closure of the polygon) we must have

$$\rho_{n+1}=\rho_1.$$

That this elimination is possible we see from the fact already mentioned, which shews that the unknowns are virtually mere unit-vectors; while each separate equation contains *coplanar* vectors only. In other words, when $\rho_m$ and $\alpha_m$ are given, $\rho_{m+1}$ is determinate without ambiguity.

We may now write the first of the equations thus:—

$$(\rho_2-\alpha_2)(\rho_1-\alpha_1)=A_1+(\alpha_1-\alpha_2)(\rho_1-\alpha_1)=q_1, \text{ suppose.}$$

Thus the angle of $q_1$ is the angle of the polygon itself, and in the same plane. By the help of the second of the above equations this becomes

$$A_2(\rho_1-\alpha_1)=(\rho_3-\alpha_2)q_1;$$

whence

$$q_2=A_2(\rho_1-\alpha_1)+(\alpha_2-\alpha_3)q_1=(\rho_3-\alpha_3)q_1.$$

By the third, this becomes

$$(\rho_4 - \alpha_3)\, q_2 = A_3 q_1 ;$$

whence

$$(\rho_4 - \alpha_4)\, q_2 = A_3 q_1 + (\alpha_3 - \alpha_4)\, q_2 = q_3.$$

The law of formation is now obvious; and, if we write

$$q_0 = \rho_1 - \alpha_1, \quad \beta_1 = \alpha_1 - \alpha_2, \quad \beta_2 = \alpha_2 - \alpha_3, \text{ \&c.,}$$

we have

$$\left.\begin{aligned} q_1 &= A_1 \quad + \beta_1 q_0 \\ q_2 &= A_2 q_0 + \beta_2 q_1 \\ q_3 &= A_3 q_1 + \beta_3 q_2 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots(1).$$

$$\text{\&c.,}$$

We have also, generally,

$$\rho_m - \alpha_m = \frac{q_{m-1}}{q_{m-2}},$$

or

$$\rho_m = \frac{q_{m-1} + \alpha_m q_{m-2}}{q_{m-2}} = \frac{A_{m-1} q_{m-3} + \alpha_{m-1} q_{m-2}}{q_{m-2}} = \frac{p_{m-2}}{q_{m-2}}, \text{ suppose} \dots (2).$$

From (1), and the value of $q_0$, we see that all the values of $q$ are *linear* functions of $\rho_1$ of the form

$$q_m = r_m + s_m \rho_1 \dots\dots\dots\dots\dots\dots\dots\dots(3).$$

By (2)
$$\begin{aligned} p_{m-1} &= A_m q_{m-2} + \alpha_m q_{m-1} \\ &= (1 + \alpha_m^2)\, q_{m-2} + \alpha_m \{A_{m-1} q_{m-3} + (\alpha_{m-1} - \alpha_m)\, q_{m-2}\} \\ &= q_{m-2} + \alpha_m (A_{m-1} q_{m-3} + \alpha_{m-1} q_{m-2}) \end{aligned}$$

$$\left.\begin{aligned} &= q_{m-2} + \alpha_m p_{m-2} \\ \text{Similarly} \quad q_{m-1} &= p_{m-2} - \alpha_m q_{m-2} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(4).$$

But the first equations in (1) give at once

$$\left.\begin{aligned} p_0 &= \quad 1 \ + \alpha_1 \rho_1 \\ q_0 &= - \alpha_1 + \quad \rho_1 \end{aligned}\right\} \quad \text{whence} \quad \left.\begin{aligned} q_0 &= + p_0 \rho_1 \\ p_0 &= - q_0 \rho_1 \end{aligned}\right\};$$

and
$$\left.\begin{aligned} p_1 &= \quad \alpha_2 - \alpha_1 + (1 + \alpha_2 \alpha_1)\, \rho_1 \\ q_1 &= 1 + \alpha_2 \alpha_1 - \quad (\alpha_2 - \alpha_1)\, \rho_1 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} q_1 &= - p_1 \rho_1 \\ p_1 &= + q_1 \rho_1 \end{aligned}\right\}.$$

This suggests that

$$\left.\begin{aligned} q_m &= (-)^m \quad p_m \rho_1 \\ p_m &= (-)^{m+1} q_m \rho_1 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots(5).$$

By (4) we have

$$p_{m-1} = q_{m-2} + \alpha_m p_{m-2},$$

$$q_{m-1} = p_{m-2} - \alpha_m q_{m-2}.$$

Let $m$ be odd, then we should have by (5)

$$p_{m-2} = A + B\rho_1,$$

$$q_{m-2} = B - A\rho_1 ;$$

whence
$$p_{m-1} = B - A\rho_1 + \alpha_m (A + B\rho_1),$$
$$q_{m-1} = A + B\rho_1 - \alpha_m (B - A\rho_1);$$
or
$$p_{m-1} = B + \alpha_m A - (A - \alpha_m B)\,\rho_1,$$
$$q_{m-1} = A - \alpha_m B + (B + \alpha_m A)\,\rho_1.$$
These agree with (5), because $m-1$ is even. And similarly we may prove the proposition when $m$ is even.

If now, in (2), we put $n+1$ for $m$, we have
$$\rho_{n+1} = \rho_1 = \frac{C + D\rho_1}{D - C\rho_1} \text{ if } n \text{ be even,}$$
$$= \frac{C - D\rho_1}{D + C\rho_1} \text{ if } n \text{ be odd,}$$
$C$ and $D$ being quaternions to be calculated (as above) from the data. The two cases require to be developed separately.

Take first the odd polygon:—

then
$$\rho_1 D + \rho_1 C\rho_1 = C - D\rho_1,$$
or
$$\rho_1 (d + \delta) + \rho_1 (c + \gamma)\,\rho_1 = c + \gamma - (d + \delta)\,\rho_1,$$
if we exhibit the scalar and vector parts of the quaternions $C$ and $D$. Cutting out the parts which cancel one another, and dividing by 2, this becomes
$$d\rho_1 + S\delta\rho_1 + \rho_1 S\gamma\rho_1 - c = 0,$$
which, as $\rho$ is finite, divides itself at once into the two equations
$$S\gamma\rho_1 + d = 0,$$
$$S\delta\rho_1 - c = 0.$$
These planes intersect in a line which, by its intersections (if real) with the sphere, gives two possible positions of the first corner of the polygon.

For the even polygon we have
$$\rho_1 D - \rho_1 C\rho_1 = C + D\rho_1,$$
or
$$V\rho_1\delta - \gamma - \rho_1 S\gamma\rho_1 = 0;$$
which may be written
$$V.\,\rho_1 (\delta - V\gamma\rho_1) = 0.$$
This equation gives, as in § 199 above,
$$\rho_1 = (x + \gamma)^{-1} \left(\frac{S\gamma\delta}{x} + \delta\right),$$
where $x$ is to be found from
$$x^2 - \gamma^2 = \frac{S^2\gamma\delta}{x^2} - \delta^2.$$

The two values of $x^2$ have opposite signs. Hence there are two real values of $x$, equal and with opposite signs, giving two real points on the sphere. Thus *this* case of the problem is always possible.

251. *To find the equation of a cone of revolution, whose vertex is the origin.*

Suppose $\alpha$, where $T\alpha = 1$, to be its axis, and $e$ the cosine of its semi-vertical angle; then, if $\rho$ be the vector of any point in the cone,

$$S\alpha U\rho = -e,$$

or

$$S^2\alpha\rho = -e^2\rho^2.$$

252. Change the origin to the point in the axis whose vector is $x\alpha$, and the equation becomes

$$(-x + S\alpha\varpi)^2 = -e^2(x\alpha + \varpi)^2.$$

Let the radius of the section of the cone made by

$$S\alpha\varpi = 0$$

retain a constant value $b$, while $x$ changes; this necessitates

$$\frac{x}{\sqrt{b^2 + x^2}} = e,$$

so that when $x$ is infinite, $e$ is unity. In this case the equation becomes

$$S^2\alpha\varpi + \varpi^2 + b^2 = 0,$$

which must therefore be the equation of a circular cylinder of radius $b$, whose axis is the vector $\alpha$. To verify this we have only to notice that if $\varpi$ be the vector of a point of such a cylinder we must (§ 214) have

$$TV\alpha\varpi = b,$$

which is the same equation as that above.

253. *To find, generally, the equation of a cone which has a circular section:—*

Take the origin as vertex, and let the circular section be the intersection of the plane

$$S\alpha\rho = 1$$

with the sphere (passing through the origin)

$$\rho^2 = S\beta\rho.$$

These equations may be written thus,

$$S\alpha U\rho = \frac{1}{T\rho},$$

$$S\beta U\rho = -T\rho.$$

Hence, eliminating $T\rho$ by multiplying the right, and left, members together, we find the following equation which $U\rho$ must satisfy—

$$S\alpha U\rho S\beta U\rho = -1,$$

or

$$\rho^2 - S\alpha\rho S\beta\rho = 0,$$

which is therefore the required equation of the cone.

As $\alpha$ and $\beta$ are similarly involved, the mere *form* of this equation proves the existence of the subcontrary section discovered by Apollonius.

254. The equation just obtained may be written

$$S.U\alpha U\rho S.U\beta U\rho = -\frac{1}{T.\alpha\beta},$$

or, since $\alpha$ and $\beta$ are perpendicular to the cyclic planes (§ 59*),

$$\sin p \sin p' = \text{constant},$$

where $p$ and $p'$ are arcs drawn from any point of a spherical conic perpendicular to the cyclic arcs. This is a well-known property of such curves.

255. If we cut the cyclic cone by any plane passing through the origin, as

$$S\gamma\rho = 0,$$

then $V\alpha\gamma$ and $V\beta\gamma$ are the traces on the cyclic planes, so that

$$\rho = xUV\alpha\gamma + yUV\beta\gamma \text{ (§ 24).}$$

Substitute in the equation of the cone, and we get

$$-x^2 - y^2 + Pxy = 0,$$

where $P$ is a known scalar. Hence the values of $x$ and $y$ are the same pair of numbers. This is a very elementary proof of the proposition in § 59*, that $\overset{\frown}{PL} = \overset{\frown}{MQ}$ (in the last figure of that section).

256. When $x$ and $y$ are *equal*, the transversal arc becomes a tangent to the spherical conic, and is evidently bisected at the point of contact. Here we have

$$P = 2 = 2S.UV\alpha\gamma UV\beta\gamma + \frac{(S.\alpha\beta\gamma)^2}{T.V\alpha\gamma V\beta\gamma}.$$

This is the equation of the cone whose sides are perpendiculars (through the origin) to the planes which touch the cyclic cone, and from this property the same equation may readily be deduced.

257. It may be well to observe that the property of the Stereographic projection of the sphere, viz. that the projection of a circle is a circle, is an immediate consequence of the above form of the equation of a cyclic cone.

258. That § 253 gives the most general form of the equation of a cone of the second degree, when the vertex is taken as origin, follows from the early results of next Chapter. For it is shewn in § 263 that the equation of a cone of the second degree can always be put in the form

$$2\Sigma \, . \, S\alpha\rho S\beta\rho + A\rho^2 = 0.$$

This may be written $S\rho\phi\rho = 0,$

where $\phi$ is the self-conjugate linear and vector function

$$\phi\rho = \Sigma V . \, \alpha\rho\beta + (A + \Sigma S\alpha\beta)\,\rho.$$

By § 180 this may be transformed to

$$\phi\rho = p\rho + V . \, \lambda\rho\mu,$$

and the general equation of the cone becomes

$$(p - S\lambda\mu)\,\rho^2 + 2S\lambda\rho\, S\mu\rho = 0,$$

which is the form obtained in § 253.

259. Taking the form

$$S\rho\phi\rho = 0$$

as the simplest, we find by differentiation

$$Sd\rho\phi\rho + S\rho d\phi\rho = 0,$$

or $$2Sd\rho\phi\rho = 0.$$

Hence $\phi\rho$ is perpendicular to the tangent-plane at the extremity of $\rho$. The equation of this plane is therefore ($\varpi$ being the vector of any point in it)

$$S\phi\rho\,(\varpi - \rho) = 0,$$

or, by the equation of the cone,

$$S\varpi\phi\rho = 0.$$

260. *The equation of the cone of normals to the tangent-planes of a given cone can be easily formed from that of the cone itself.* For we may write the equation of the cone in the form

$$S . \, \phi\rho\phi^{-1}\phi\rho = 0;$$

and if we put $\phi\rho = \sigma$, a vector of the new cone, the equation becomes

$$S\sigma\phi^{-1}\sigma = 0.$$

Numerous curious properties of these connected cones, and of the corresponding spherical conics, follow at once from these equations. But we must leave them to the reader.

261. As a final example, let us *find the equation of a cyclic cone when five of its vector-sides are given*—i.e. *find the cone of the second degree whose vertex is the origin, and on whose surface lie the vectors* $\alpha, \beta, \gamma, \delta, \epsilon$.

If we write, after Hamilton,

$$0 = S.V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\rho)V(V\gamma\delta V\rho\alpha) \ldots\ldots\ldots (1)$$

we have the equation of a *cone* whose vertex is the origin—for the equation is not altered by putting $x\rho$ for $\rho$. Also it is the equation of a cone of the second degree, since $\rho$ occurs only twice. Moreover the vectors $\alpha, \beta, \gamma, \delta, \epsilon$ are sides of the cone, because if any one of them be put for $\rho$ the equation is satisfied. Thus if we put $\beta$ for $\rho$ the equation becomes

$$\begin{aligned} 0 &= S.V(V\alpha\beta V\delta\epsilon)V(V\beta\gamma V\epsilon\beta)V(V\gamma\delta V\beta\alpha) \\ &= S.V(V\alpha\beta V\delta\epsilon)\{V\beta\alpha S.V\gamma\delta V\beta\gamma V\epsilon\beta - V\gamma\delta S.V\beta\alpha V\beta\gamma V\epsilon\beta\}. \end{aligned}$$

The first term vanishes because

$$S.V(V\alpha\beta V\delta\epsilon)V\beta\alpha = 0,$$

and the second because

$$S.V\beta\alpha V\beta\gamma V\epsilon\beta = 0,$$

since the three vectors $V\beta\alpha$, $V\beta\gamma$, $V\epsilon\beta$, being each at right angles to $\beta$, must be in one plane.

As is remarked by Hamilton, this is a very simple proof of Pascal's Theorem—for (1) is the condition that the intersections of the planes of $\alpha, \beta$ and $\delta, \epsilon$; $\beta, \gamma$ and $\epsilon, \rho$; $\gamma, \delta$ and $\rho, \alpha$; shall lie in one plane; or, making the statement for any plane section of the cone:—*In order that the points of intersection of the three pairs of opposite sides, of a hexagon inscribed in a curve, may always lie in one straight line, the curve must be a conic section.*

## EXAMPLES TO CHAPTER VIII.

1. On the vector of a point $P$ in the plane

$$S\alpha\rho = 1$$

a point $Q$ is taken, such that $QO \,.\, OP$ is constant; find the equation of the locus of $Q$.

2. What spheres cut the loci of $P$ and $Q$ in (1) so that both circles of intersection lie on a cone whose vertex is $O$?

3. A sphere touches a fixed plane, and cuts a fixed sphere. If the point of contact with the plane be given, the plane of the intersection of the spheres contains a fixed line.

Find the locus of the centre of the variable sphere, if the plane of its intersection with the fixed sphere passes through a given point.

4. Find the radii of the spheres which touch, simultaneously, the four given planes

$$S\alpha\rho = 0, \qquad S\beta\rho = 0, \qquad S\gamma\rho = 0, \qquad S\delta\rho = 1.$$

[What is the volume of the tetrahedron enclosed by these planes?]

5. If a moveable line, passing through the origin, make with any number of fixed lines angles $\theta$, $\theta_1$, $\theta_2$, &c., such that

$$a \cos\theta + a_1 \cos\theta_1 + \ldots\ldots = \text{constant},$$

where $a$, $a_1$,......are constant scalars, the line describes a right cone.

6. Determine the conditions that

$$S\rho\phi\rho = 0$$

may represent a *right* cone, $\phi$ being as in § 258.

7. What property of a cone (or of a spherical conic) is given directly by the following particular form of its equation,

$$S \,.\, \iota\rho\kappa\rho = 0\,?$$

8. What are the conditions that the surfaces represented by

$$S\rho\phi\rho = 0, \quad \text{and} \quad S \,.\, \iota\rho\kappa\rho = 0,$$

may degenerate into pairs of planes?

9. If arcs of great circles, drawn from any given point of a sphere to a fixed great circle, be bisected, find the locus of these middle points; and shew that the arcs drawn from the pole of this fixed great circle, to that of which the given point is pole, are also bisected by the same locus.

10. Find the locus of the vertices of all right cones which have a common ellipse as base.

11. Two right circular cones have their axes parallel. Find the orthogonal projection of their curve of intersection on the plane containing their axes.

12. Two spheres being given in magnitude and position, every sphere which intersects them in given angles will touch two other fixed spheres and cut a third at right angles.

13. If a sphere be placed on a table, the breadth of the elliptic shadow formed by rays diverging from a fixed point is independent of the position of the sphere.

14. Form the equation of the cylinder which has a given circular section, and a given axis. Find the direction of the normal to the subcontrary section.

15. Given the base of a spherical triangle, and the product of the cosines of the sides, the locus of the vertex is a spherical conic, the poles of whose cyclic arcs are the extremities of the given base.

16. (Hamilton, *Bishop Law's Premium Ex.*, 1858.)

(*a*) What property of a sphero-conic is most immediately indicated by the equation

$$S\frac{\rho}{\alpha}S\frac{\beta}{\rho}=1\,?$$

(*b*) The equation

$$(V\lambda\rho)^2+(S\mu\rho)^2=0$$

also represents a cone of the second order; $\lambda$ is a focal line, and $\mu$ is perpendicular to the director-plane corresponding.

(*c*) What property of a sphero-conic does the equation most immediately indicate ?

17. Shew that the areas of all triangles, bounded by a tangent to a spherical conic and by the cyclic arcs, are equal.

18. Shew that the locus of a point, the sum of whose areual distances from two given points on a sphere is constant, is a spherical conic.

19. If two tangent planes be drawn to a cyclic cone, the four lines in which they intersect the cyclic planes are sides of a right cone.

20. Find the equation of the cone whose sides are the intersections of pairs of mutually perpendicular tangent planes to a given cyclic cone.

21. Find the condition that five given points may lie on a sphere.

22. What is the surface denoted by the equation

$$\rho^2 = x\alpha^2 + y\beta^2 + z\gamma^2,$$

where

$$\rho = x\alpha + y\beta + z\gamma,$$

$\alpha$, $\beta$, $\gamma$ being given vectors, and $x$, $y$, $z$ variable scalars?

Express the equation of the surface in terms of $\rho$, $\alpha$, $\beta$, $\gamma$ alone.

23. Find the equation of the cone whose sides bisect the angles between a fixed line, and any line in a given plane, which meets the fixed line.

What property of a spherical conic is most directly given by this result?

# CHAPTER IX.

## SURFACES OF THE SECOND DEGREE.

262. THE general scalar equation of the second degree in a vector $\rho$ must evidently contain a term independent of $\rho$, terms of the form $S . a\rho b$ involving $\rho$ to the first degree, and others of the form $S . a\rho b\rho c$ involving $\rho$ to the second degree, $a$, $b$, $c$, &c. being constant quaternions. Now the term $S . a\rho b$ may be written as

$$S\rho V(ba),$$

or as

$$S . (Sa + Va)\,\rho\,(Sb + Vb) = SaS\rho Vb + SbS\rho Va + S . \rho VbVa,$$

each of which may evidently be put in the form $S\gamma\rho$, where $\gamma$ is a known vector.

Similarly* the term $S . a\rho b\rho c$ may be reduced to a set of terms, each of which has one of the forms

$$A\rho^2, \quad (S\alpha\rho)^2, \quad S\alpha\rho S\beta\rho,$$

the second being merely a particular case of the third. Thus (the numerical factors 2 being introduced for convenience) we may write the general scalar equation of the second degree as follows:—

$$2\Sigma . S\alpha\rho S\beta\rho + A\rho^2 + 2S\gamma\rho = C \ldots\ldots\ldots\ldots\ldots (1).$$

263. Change the origin to $D$ where $\overline{OD} = \delta$, then $\rho$ becomes $\rho + \delta$, and the equation takes the form

$$2\Sigma . S\alpha\rho S\beta\rho + A\rho^2 + 2\Sigma\,(S\alpha\rho S\beta\delta + S\beta\rho S\alpha\delta) + 2AS\delta\rho + 2S\gamma\rho$$
$$+ 2\Sigma . S\alpha\delta S\beta\delta + A\delta^2 + 2S\gamma\delta - C = 0\,;$$

---

* For $S . a\rho b\rho c = S . ca\rho b\rho = S . a'\rho b\rho = (2Sa'Sb - Sa'b)\,\rho^2 + 2Sa'\rho Sb\rho$ ; and in particular cases we may have $Va' = Vb$.

from which the first power of $\rho$ disappears, that is *the surface is referred to its centre,* if

$$\Sigma(\alpha S\beta\delta + \beta S\alpha\delta) + A\delta + \gamma = 0 \ldots\ldots\ldots\ldots\ldots(2),$$

a vector equation of the first degree, which in general gives a single definite value for $\delta$, by the processes of Chapter V. [It would lead us beyond the limits of an elementary treatise to consider the special cases in which (2) represents a line, or a plane, any point of which is a centre of the surface. The processes to be employed in such special cases have been amply illustrated in the Chapter referred to.]

With this value of $\delta$, and putting

$$D = C - 2S\gamma\delta - A\delta^2 - 2\Sigma . S\alpha\delta S\beta\delta,$$

the equation becomes

$$2\Sigma . S\alpha\rho S\beta\rho + A\rho^2 = D.$$

If $D = 0$, the surface is conical (a case treated in last Chapter); if not, it is an ellipsoid or hyperboloid. Unless expressly stated not to be, the surface will, when $D$ is not zero, be considered an ellipsoid. By this we avoid for the time some rather delicate considerations.

By dividing by $-D$, and thus altering only the tensors of the constants, we see that the equation of central surfaces of the second degree, referred to the centre, is (excluding cones)

$$2\Sigma(S\alpha\rho S\beta\rho) + g\rho^2 = -1 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

[It is convenient to use the negative sign in the right-hand member, as this ensures that the important vector $\phi\rho$ (which we must soon introduce) shall make an acute angle with $\rho$; i.e. be drawn, on the whole, towards the same parts.]

264. Differentiating, we obtain

$$2\Sigma\{S\alpha d\rho S\beta\rho + S\alpha\rho S\beta d\rho\} + 2gS\rho d\rho = 0,$$

or $$S . d\rho\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 0,$$

and therefore, by § 144, the tangent plane is

$$S(\varpi - \rho)\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = 0,$$

i.e. $$S . \varpi\{\Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho\} = -1, \text{ by (3).}$$

Hence, if $$\nu = \Sigma(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho \ldots\ldots\ldots\ldots\ldots (4),$$

the tangent plane is $$S\nu\varpi = -1,$$

and the surface itself is $$S\nu\rho = -1.$$

And, as $-\nu^{-1}$ (being perpendicular to the tangent plane, and satisfying its equation) is evidently the vector-perpendicular from the origin on the tangent plane, $\nu$ is called the *vector of proximity*.

265. Hamilton uses for $\nu$, which is obviously a linear and vector function of $\rho$, the notation $\phi\rho$, $\phi$ expressing a functional operator, as in Chapter V. But, for the sake of clearness, we will go over part of the ground again, especially in the interests of students who have mastered only the more elementary parts of that Chapter.

We have, then, $\phi\rho = \Sigma\,(\alpha S\beta\rho + \beta S\alpha\rho) + g\rho.$

With this definition of $\phi$, it is easy to see that

(*a*) $\phi\,(\rho + \sigma) = \phi\rho + \phi\sigma$, &c., for *any* two or more vectors.

(*b*) $\phi\,(x\rho) = x\phi\rho$, a particular case of (*a*), $x$ being a scalar.

(*c*) $d\phi\rho = \phi\,(d\rho)$.

(*d*) $S\sigma\phi\rho = \Sigma\,(S\alpha\sigma S\beta\rho + S\beta\sigma S\alpha\rho) + gS\rho\sigma = S\rho\phi\sigma$,

or $\phi$ is, in this case, self-conjugate.

This last property is of great importance in what follows.

266. Thus the general equation of central surfaces of the second degree (excluding cones) may now be written

$$S\rho\phi\rho = -1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Differentiating, $$Sd\rho\phi\rho + S\rho d\phi\rho = 0,$$

which, by applying (*c*) and then (*d*) to the last term on the left, gives

$$2S\phi\rho d\rho = 0,$$

and therefore, as in § 264, though now much more simply, the tangent plane at the extremity of $\rho$ is

$$S\,(\varpi - \rho)\,\phi\rho = 0,$$

or $$S\varpi\phi\rho = S\rho\phi\rho = -1.$$

If this pass through $A$ ($\overline{OA} = \alpha$), we have

$$S\alpha\phi\rho = -1,$$

or, by (*d*), $$S\rho\phi\alpha = -1,$$

for all possible points of contact.

This is therefore the equation of the plane of contact of tangent planes drawn from $A$.

267. *To find the enveloping cone whose vertex is $A$*, notice that

$$(S\rho\phi\rho + 1) + p\,(S\rho\phi\alpha + 1)^2 = 0,$$

where $p$ is any scalar, is the equation of a surface of the second degree *touching* the ellipsoid along its intersection with the plane. If this pass through $A$ we have

$$(S\alpha\phi\alpha + 1) + p\,(S\alpha\phi\alpha + 1)^2 = 0,$$

and $p$ is found. Then our equation becomes

$$(S\rho\phi\rho + 1)\,(S\alpha\phi\alpha + 1) - (S\rho\phi\alpha + 1)^2 = 0 \ldots\ldots\ldots \quad (1),$$

which is the cone required. To assure ourselves of this, transfer the origin to $A$, by putting $\rho + \alpha$ for $\rho$. The result is, using $(a)$ and $(d)$,

$$(S\rho\phi\rho + 2S\rho\phi\alpha + S\alpha\phi\alpha + 1)\,(S\alpha\phi\alpha + 1) - (S\rho\phi\alpha + S\alpha\phi\alpha + 1)^2 = 0,$$

or
$$S\rho\phi\rho\,(S\alpha\phi\alpha + 1) - (S\rho\phi\alpha)^2 = 0,$$

which is homogeneous in $T\rho$, and is therefore the equation of a cone.

[In the special case when $A$ lies *on* the surface, we have

$$S\alpha\phi\alpha + 1 = 0,$$

and the value of $p$ is infinite. But this is not a case of failure, for the enveloping cone degenerates into the tangent plane

$$S\rho\phi\alpha + 1 = 0.]$$

Suppose $A$ infinitely distant, then we may put in (1) $x\alpha$ for $\alpha$, where $x$ is infinitely great, and, omitting all but the higher terms, the equation of the cylinder formed by tangent lines parallel to $\alpha$ is

$$(S\rho\phi\rho + 1)\,S\alpha\phi\alpha - (S\rho\phi\alpha)^2 = 0.$$

See, on this matter, Ex. 21 at end of Chapter.

268 To study the nature of the surface more closely, let us *find the locus of the middle points of a system of parallel chords.*

Let them be parallel to $\alpha$, then, if $\varpi$ be the vector of the middle point of one of them, $\varpi + x\alpha$ and $\varpi - x\alpha$ are values of $\rho$ which ought simultaneously to satisfy (1) of § 266.

That is
$$S\,.\,(\varpi \pm x\alpha)\,\phi\,(\varpi \pm x\alpha) = -1.$$

Hence, by $(a)$ and $(d)$, as before,

$$S\varpi\phi\varpi + x^2 S\alpha\phi\alpha = -1,$$

$$S\varpi\phi\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The latter equation shews that the locus of the extremity of $\varpi$, the middle point of a chord parallel to $\alpha$, is a plane through the centre, whose normal is $\phi\alpha$; that is, a plane parallel to the tangent plane at the point where $OA$ cuts the surface. And ($d$) shews that this relation is reciprocal—so that if $\beta$ be *any* value of $\varpi$, i.e. be any vector in the plane (1), $\alpha$ will be a vector in a diametral plane which bisects all chords parallel to $\beta$. The equations of these planes are

$$S\varpi\phi\alpha = 0,$$

$$S\varpi\phi\beta = 0,$$

so that if $V.\,\phi\alpha\phi\beta = \gamma$ (suppose) is their line of intersection, we have

and (1) gives

$$\left.\begin{array}{r} S\gamma\phi\alpha = 0 = S\alpha\phi\gamma \\ S\gamma\phi\beta = 0 = S\beta\phi\gamma \\ S\beta\phi\alpha = 0 = S\alpha\phi\beta \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence there is *an infinite number of sets of three vectors* $\alpha$, $\beta$, $\gamma$, *such that all chords parallel to any one are bisected by the diametral plane containing the other two.*

269. It is evident from § 23 that any vector may be expressed as a linear function of any three others not in the same plane; let then

$$\rho = x\alpha + y\beta + z\gamma,$$

where, by last section,

$$S\alpha\phi\beta = S\beta\phi\alpha = 0,$$

$$S\alpha\phi\gamma = S\gamma\phi\alpha = 0,$$

$$S\beta\phi\gamma = S\gamma\phi\beta = 0.$$

And let

$$\left.\begin{array}{r} S\alpha\phi\alpha = -1 \\ S\beta\phi\beta = -1 \\ S\gamma\phi\gamma = -1 \end{array}\right\},$$

so that $\alpha$, $\beta$, and $\gamma$ are vector conjugate semi-diameters of the surface we are engaged on.

Substituting the above value of $\rho$ in the equation of the surface, and attending to the equations in $\alpha$, $\beta$, $\gamma$ and to ($a$), ($b$), and ($d$), we have

$$\begin{aligned} S\rho\phi\rho &= S(x\alpha + y\beta + z\gamma)\,\phi\,(x\alpha + y\beta + z\gamma), \\ &= -(x^2 + y^2 + z^2) = -1. \end{aligned}$$

To transform this equation to Cartesian cöordinates, we notice that $x$ is the ratio which the projection of $\rho$ on $\alpha$ bears to $\alpha$ itself, &c.

If therefore we take the conjugate diameters as axes of $\xi$, $\eta$, $\zeta$, and their lengths as $a$, $b$, $c$, the above equation becomes at once

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1,$$

the ordinary equation of the ellipsoid referred to conjugate diameters.

270. If we write $\psi^2$ instead of $\phi$, these equations assume an interesting form. We take for granted, what we shall afterwards prove, that this extraction of the square root of the vector function is lawful, and that the new linear and vector function has the same properties ($a$), ($b$), ($c$), ($d$) (§ 265) as the old. The equation of the surface now becomes

$$S\rho\psi^2\rho=-1,$$

or $$S\psi\rho\psi\rho=-1,$$

or, finally, $$T\psi\rho=1.$$

If we compare this with the equation of the unit-sphere

$$T\rho=1,$$

we see at once the analogy between the two surfaces. *The sphere can be changed into the ellipsoid, or vice versâ, by a linear deformation of each vector, the operator being the function* $\psi$ *or its inverse.* See the Chapter on Kinematics.

271. Equations (2) § 268, by § 270 become

$$S\alpha\psi^2\beta=0=S\psi\alpha\psi\beta, \text{ \&c.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

so that $\psi\alpha$, $\psi\beta$, $\psi\gamma$, *the vectors of the unit-sphere which correspond to semi-conjugate diameters of the ellipsoid, form a rectangular system.*

We may remark here, that, as the equation of the ellipsoid referred to its principal axes is a case of § 269, we may now suppose $i$, $j$, and $k$ to have these directions, and the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which, in quaternions, is

$$S\rho\phi\rho=-\frac{(Si\rho)^2}{a^2}-\frac{(Sj\rho)^2}{b^2}-\frac{(Sk\rho)^2}{c^2}=-1.$$

We here tacitly assume the existence of such axes, but in all cases, by the help of Hamilton's method, developed in Chapter V., we at once arrive at the cubic equation which gives them.

It is evident from the last-written equation that

$$\phi\rho = -\left(\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}\right),$$

and
$$\psi\rho = -\left(\frac{iSi\rho}{a} + \frac{jSj\rho}{b} + \frac{kSk\rho}{c}\right.,$$

which latter may be easily proved by shewing that

$$\psi^2\rho = \phi\rho.$$

And this expression enables us to verify the assertion of last section about the properties of $\psi$.

As $Si\rho = -x$, &c., $x$, $y$, $z$ being the Cartesian cöordinates referred to the principal axes, we have now the means of at once transforming any quaternion result connected with the ellipsoid into the ordinary one.

272. Before proceeding to other forms of the equation of the ellipsoid, we may use those already given in solving a few problems.

*Find the locus of a point when the perpendicular from the centre on its polar plane is of constant length.*

If $\varpi$ be the vector of the point, the polar plane is

$$S\rho\phi\varpi = -1,$$

and the length of the perpendicular from $O$ is $\dfrac{1}{T\phi\varpi}$ (§ 264).

Hence the required locus is

$$T\phi\varpi = C,$$

or
$$S\varpi\phi^2\varpi = -C^2,$$

a concentric ellipsoid, with its axes in the same directions as those of the first. By § 271 its Cartesian equation is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = C^2.$$

273. *Find the locus of a point whose distance from a given point is always in a given ratio to its distance from a given line*

Let $\rho = x\beta$ be the given line, and $A$ $(\overline{OA} = \alpha)$ the given point, and choose the origin so that $S\alpha\beta = 0$. Then for any one of the required points

$$T(\rho - \alpha) = eTV\beta\rho.$$

This is the equation of a surface of the second degree, which may be written

$$\rho^2 - 2S\alpha\rho + \alpha^2 = e^2(S^2\beta\rho - \beta^2\rho^2).$$

Let the centre be at $\delta$, and make it the origin, then

$$\rho^2 + 2S\rho\,(\delta - \alpha) + (\delta - \alpha)^2 = e^2\,\{S^2.\,\beta\,(\rho + \delta) - \beta^2\,(\rho + \delta)^2\},$$

and, that the first power of $\rho$ may disappear,

$$(\delta - \alpha) = e^2\,(\beta S\beta\delta - \beta^2\delta),$$

a linear equation for $\delta$. To solve it, note that $S\alpha\beta = 0$; operate by $S.\beta$, and we get

$$(1 - e^2\beta^2 + e^2\beta^2)\,S\beta\delta = S\beta\delta = 0.$$

Hence $$\delta - \alpha = -\,e^2\beta^2\delta,$$

or $$\delta = \frac{\alpha}{1 + e^2\beta^2}.$$

Referred to this point as origin the equation becomes

$$(1 + e^2\beta^2)\,\rho^2 - e^2 S^2\beta\rho + \frac{e^2\beta^2\alpha^2}{1 + e^2\beta^2} = 0,$$

which shews that it belongs to a surface of revolution (of the second degree) whose axis is parallel to $\beta$, since its intersection with a plane $S\beta\rho = a$, perpendicular to that axis, lies also on the sphere

$$\rho = \frac{e^2a^2}{1 + e^2\beta^2} - \frac{e^2\beta^2\alpha^2}{(1 + e^2\beta^2)^2}.$$

In fact, if the point be the focus of any meridian section of an oblate spheroid, the line is the directrix of the same.

274. *A sphere, passing through the centre of an ellipsoid, is cut by a series of spheres whose centres are on the ellipsoid and which pass through the centre thereof; find the envelop of the planes of intersection.*

Let $(\rho - \alpha)^2 = \alpha^2$ be the first sphere, i.e.

$$\rho^2 - 2S\alpha\rho = 0.$$

One of the others is $$\rho^2 - 2S\varpi\rho = 0,$$

where $$S\varpi\phi\varpi = -1.$$

The plane of intersection is

$$S\,(\varpi - \alpha)\,\rho = 0.$$

Hence, for the envelop (see next Chapter),

$$\left.\begin{aligned} S\varpi'\phi\varpi &= 0 \\ S\varpi'\rho &= 0 \end{aligned}\right\},\ \text{where } \varpi' = d\varpi,$$

or $$\phi\varpi = x\rho, \quad \{Vx = 0\},$$

i.e. $$\varpi = x\phi^{-1}\rho.$$

Hence and
$$\left.\begin{array}{l} x^2 S\rho\phi^{-1}\rho = -1 \\ x S\rho\phi^{-1}\rho = S\alpha\rho \end{array}\right\},$$

and, eliminating $x$,
$$S\rho\phi^{-1}\rho = -(S\alpha\rho)^2,$$

a cone of the second degree.

275. *From a point in the outer of two concentric ellipsoids a tangent cone is drawn to the inner, find the envelop of the plane of contact.*

If $S\varpi\phi\varpi = -1$ be the outer, and $S\rho\psi\rho = -1$ be the inner, $\phi$ and $\psi$ being any two self-conjugate linear and vector functions, the plane of contact is
$$S\varpi\psi\rho = -1.$$

Hence, for the envelop,
$$\left.\begin{array}{l} S\varpi'\psi\rho = 0 \\ S\varpi'\phi\varpi = 0 \end{array}\right\},$$

therefore
$$\phi\varpi = x\psi\rho,$$

or
$$\varpi = x\phi^{-1}\psi\rho.$$

This gives and
$$\left.\begin{array}{l} xS.\psi\rho\phi^{-1}\psi\rho = -1 \\ x^2 S.\psi\rho\phi^{-1}\psi\rho = -1 \end{array}\right\},$$

and therefore, eliminating $x$,
$$S.\psi\rho\phi^{-1}\psi\rho = -1,$$

or
$$S.\rho\psi\phi^{-1}\psi\rho = -1,$$

another concentric ellipsoid, as $\psi\phi^{-1}\psi$ is a linear and vector function $=\chi$ suppose; so that the equation may be written
$$S\rho\chi\rho = -1.$$

276. *Find the locus of intersection of tangent planes at the extremities of conjugate diameters.*

If $\alpha$, $\beta$, $\gamma$ be the vector semi-diameters, the planes are
$$\left.\begin{array}{l} S\varpi\psi^2\alpha = -1 \\ S\varpi\psi^2\beta = -1 \\ S\varpi\psi^2\gamma = -1 \end{array}\right\},$$

with the conditions § 271.

Hence $-\psi\varpi S.\psi\alpha\psi\beta\psi\gamma = \psi\varpi = \psi\alpha + \psi\beta + \psi\gamma$, by § 92,

therefore
$$T\psi\varpi = \sqrt{3},$$

since $\psi\alpha$, $\psi\beta$, $\psi\gamma$ form a rectangular system of unit-vectors.

This may also evidently be written
$$S\varpi\psi^2\varpi = -3,$$

shewing that the locus is similar and similarly situated to the given ellipsoid, but larger in the ratio $\sqrt{3} : 1$.

277. *Find the locus of intersection of three tangent planes mutually at right angles.*

If $\rho$ be the point of contact,

$$S\varpi\psi^2\rho = -1$$

is the equation of the tangent plane.

The vector perpendicular from the origin is (§ 264)

$$-\frac{1}{\psi^2\rho} = x\alpha, \text{ suppose,}$$

where $\alpha$ is a unit-vector. This gives

$$\psi^2\rho = \alpha/x;$$

whence, by the equation of the ellipsoid,

$$\frac{1}{x^2}S\alpha\psi^{-2}\alpha = S\rho\psi^2\rho = -1,$$

Thus the perpendicular is

$$\alpha\sqrt{-S\alpha\psi^{-2}\alpha} = \alpha T\psi^{-1}\alpha.$$

The sum of the squares of these, corresponding to a rectangular unit-system, is

$$-\Sigma S\alpha\psi^{-2}\alpha = m_2,$$

by § 185. See also § 279.

278. *Find the locus of the intersection of three spheres whose diameters are semi-conjugate diameters of an ellipsoid.*

If $\alpha$ be one of the semi-conjugate diameters

$$S\alpha\psi^2\alpha = -1.$$

And the corresponding sphere is

$$\rho^2 - S\alpha\rho = 0,$$

or

$$\rho^2 - S\psi\alpha\psi^{-1}\rho = 0,$$

with similar equations in $\beta$ and $\gamma$. Hence, by § 92,

$$\psi^{-1}\rho S . \psi\alpha\psi\beta\psi\gamma = -\psi^{-1}\rho = \rho^2(\psi\alpha + \psi\beta + \psi\gamma),$$

and, taking tensors, $T\psi^{-1}\rho = \sqrt{3}T\rho^2$,

or $T\psi^{-1}\rho^{-1} = \sqrt{3}$,

or, finally, $S\rho\psi^{-2}\rho = -3\rho^4$.

This is Fresnel's *Surface of Elasticity*, Chap. XII.

279. Before going farther we may prove some useful properties of the function $\phi$ in the form we are at present using—viz.

$$\phi\rho = -\left(\frac{iSi\rho}{a^2} + \frac{jSj\rho}{b^2} + \frac{kSk\rho}{c^2}\right).$$

We have $$\rho = -iSi\rho - jSj\rho - kSk\rho,$$

and it is evident that

$$\phi i = \frac{i}{a^2}, \quad \phi j = \frac{j}{b^2}, \quad \phi k = \frac{k}{c^2}.$$

Hence $$\phi^2\rho = -\left(\frac{iSi\rho}{a^4} + \frac{jSj\rho}{b^4} + \frac{kSk\rho}{c^4}\right).$$

Also $$\phi^{-1}\rho = -(a^2 iSi\rho + b^2 jSj\rho + c^2 kSk\rho),$$

and so on.

Again, if $\alpha$, $\beta$, $\gamma$ be *any* rectangular unit-vectors

$$S\alpha\phi\alpha = -\left(\frac{(Si\alpha)^2}{a^2} + \frac{(Sj\alpha)^2}{b^2} + \frac{(Sk\alpha)^2}{c^2}\right),$$

$$\&c. = \&c.$$

But as $$(Si\rho)^2 + (Sj\rho)^2 + (Sk\rho)^2 = -\rho^2,$$

we have $$S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma = -\left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right\}.$$

Similarly

$$S\alpha\phi^2\alpha + S\beta\phi^2\beta + S\gamma\phi^2\gamma = (\phi\alpha)^2 + (\phi\beta)^2 + (\phi\gamma)^2 = -\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right).$$

Again,

$$S.\phi\alpha\phi\beta\phi\gamma = -S.\left(\frac{iSi\alpha}{a^2} + \dots\right)\left(\frac{iSi\beta}{a^2} + \dots\right)\left(\frac{iSi\gamma}{a^2} + \dots\right)$$

$$= \begin{vmatrix} \frac{Si\alpha}{a^2}, & \frac{Sj\alpha}{b^2}, & \frac{Sk\alpha}{c^2} \\ \frac{Si\beta}{a^2}, & \frac{Sj\beta}{b^2}, & \frac{Sk\beta}{c^2} \\ \frac{Si\gamma}{a^2}, & \frac{Sj\gamma}{b^2}, & \frac{Sk\gamma}{c^2} \end{vmatrix} = \frac{1}{a^2b^2c^2} \begin{vmatrix} Si\alpha, & Sj\alpha, & Sk\alpha \\ Si\beta, & Sj\beta, & Sk\beta \\ Si\gamma, & Sj\gamma, & Sk\gamma \end{vmatrix} = -\frac{1}{a^2b^2c^2}.$$

And so on. These elementary investigations are given here for the benefit of those who have not read Chapter V. The student may easily obtain all such results in a far more simple manner by means of the formulae of that Chapter.

280. *Find the locus of intersection of a rectangular system of three tangents to an ellipsoid.*

If $\varpi$ be the vector of the point of intersection, $\alpha$, $\beta$, $\gamma$ the tangents, then, since $\varpi + x\alpha$ must give equal values of $x$ when substituted in the equation of the surface, so that

$$S(\varpi + x\alpha)\,\phi(\varpi + x\alpha) = -1,$$

or

$$x^2 S\alpha\phi\alpha + 2xS\varpi\phi\alpha + (S\varpi\phi\varpi + 1) = 0,$$

we have

$$(S\varpi\phi\alpha)^2 = S\alpha\phi\alpha\,(S\varpi\phi\varpi + 1).$$

Adding this to the two similar equations in $\beta$ and $\gamma$, we have

$$(S\alpha\phi\varpi)^2 + (S\beta\phi\varpi)^2 + (S\gamma\phi\varpi)^2 = (S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma)\,(S\varpi\phi\varpi + 1),$$

or

$$(\phi\varpi)^2 = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)(S\varpi\phi\varpi + 1),$$

or

$$S\,.\,\varpi\left\{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\phi - \phi^2\right\}\varpi = -\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right),$$

an ellipsoid concentric with the first.

281. *If a rectangular system of chords be drawn through any point within an ellipsoid, the sum of the reciprocals of the rectangles under the segments into which they are divided is constant.*

With the notation of the solution of the preceding problem, $\varpi$ giving the intersection of the vectors, it is evident that the product of the values of $x$ is one of the rectangles in question taken negatively.

Hence the required sum is

$$-\frac{\Sigma S\alpha\phi\alpha}{S\varpi\phi\varpi + 1} = +\frac{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}}{S\varpi\phi\varpi + 1}.$$

This evidently depends on $S\varpi\phi\varpi$ only and not on the particular directions of $\alpha$, $\beta$, $\gamma$: and is therefore unaltered if $\varpi$ be the vector of any point of an ellipsoid similar, and similarly situated, to the given one. [The expression is interpretable even if the point be exterior to the ellipsoid.]

282. *Shew that if any rectangular system of three vectors be drawn from a point of an ellipsoid, the plane containing their other extremities passes through a fixed point. Find the locus of the latter point as the former varies.*

With the same notation as before, we have

$$S\varpi\phi\varpi = -1,$$

and

$$S(\varpi + x\alpha)\,\phi(\varpi + x\alpha) = -1;$$

therefore

$$x = -\frac{2S\alpha\phi\varpi}{S\alpha\phi\alpha}.$$

Hence the required plane passes through the extremity of

$$\varpi - 2\alpha\frac{S\alpha\phi\varpi}{S\alpha\phi\alpha},$$

and those of two other vectors similarly determined. It therefore (see § 30) passes through the point whose vector is

$$\theta = \varpi - 2\frac{\alpha S\alpha\phi\varpi + \beta S\beta\phi\varpi + \gamma S\gamma\phi\varpi}{S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma},$$

or

$$\theta = \varpi - \frac{2\phi\varpi}{m_2} \quad (\S\ 185).$$

Thus the first part of the proposition is proved.

But we have also $\varpi = -\frac{m_2}{2}\left(\phi - \frac{m_2}{2}\right)^{-1}\theta,$

whence by the equation of the ellipsoid we obtain

$$\frac{m_2^{\,2}}{4}S\,.\,\theta\left(\phi - \frac{m_2}{2}\right)^{-1}\phi\left(\phi - \frac{m_2}{2}\right)^{-1}\theta = -1,$$

the equation of a concentric ellipsoid.

283. *Find the directions of the three vectors which are parallel to a set of conjugate diameters in each of two central surfaces of the second degree.*

Transferring the centres of both to the origin, let their equations be

$$\left.\begin{aligned} S\rho\phi\rho &= -1 \text{ or } 0 \\ \text{and} \qquad S\rho\psi\rho &= -1 \text{ or } 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If $\alpha, \beta, \gamma$ be vectors in the required directions, we must have (§ 268)

$$\left.\begin{aligned} S\alpha\phi\beta &= 0, \qquad & S\alpha\psi\beta &= 0 \\ S\beta\phi\gamma &= 0, & S\beta\psi\gamma &= 0 \\ S\gamma\phi\alpha &= 0, & S\gamma\psi\alpha &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From these equations $\phi\alpha \parallel V\beta\gamma \parallel \psi\alpha$, &c.

Hence the three required directions are the roots of

$$V.\,\phi\rho\psi\rho = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This is evident on other grounds, for it means that *if one of the surfaces expand or contract uniformly till it meets the other, it will touch it successively at points on the three sought vectors.*

We may put (3) in either of the following forms—

$$\left.\begin{aligned} & V.\,\rho\phi^{-1}\psi\rho = 0 \\ \text{or} \qquad & V.\,\rho\psi^{-1}\phi\rho = 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and, as $\phi$ and $\psi$ are given functions, we find the solutions (when they are all real, so that the problem is possible) by the processes of Chapter V.

[*Note.* As $\phi^{-1}\psi$ and $\psi^{-1}\phi$ are not, in general, self-conjugate functions, equations (4) do not signify that $\alpha$, $\beta$, $\gamma$ are vectors parallel to the principal axes of the surfaces

$$S.\rho\phi^{-1}\psi\rho=-1, \qquad S.\rho\psi^{-1}\phi\rho=-1.$$

In *these* equations it does not matter whether $\phi^{-1}\psi$ is self-conjugate or not; but it does most particularly matter when, as in (4), they are involved in such a manner that their non-conjugate parts do not vanish.]

*Given two surfaces of the second degree, which have parallel conjugate diameters, every surface of the second degree passing through their intersection has conjugate diameters parallel to these.*

For any surface of the second degree through the intersection of

$$S\rho\phi\rho=-1 \quad \text{and} \quad S(\rho-\alpha)\psi(\rho-\alpha)=-e,$$

is
$$fS\rho\phi\rho-S(\rho-\alpha)\psi(\rho-\alpha)=e-f,$$

where $e$ and $f$ are scalars, of which $f$ is variable.

The axes of this depend only on the term

$$S\rho(f\phi-\psi)\rho.$$

Hence the set of conjugate diameters which are the same in all are parallel to the roots of

$$V(f\phi-\psi)\rho(f_1\phi-\psi)\rho=0, \quad \text{or} \quad V\phi\rho\psi\rho=0,$$

as we might have seen without analysis.

The locus of the centres is given by the equation

$$(\psi-f\phi)\rho-\psi\alpha=0,$$

where $f$ is a scalar variable.

284. *Find the equation of the ellipsoid of which three conjugate semi-diameters are given.*

Let the vector semi-diameters be $\alpha$, $\beta$, $\gamma$, and let

$$S\rho\phi\rho=-1$$

be the equation of the ellipsoid. Then (§ 269) we have

$$\begin{aligned} S\alpha\phi\alpha&=-1, \qquad & S\alpha\phi\beta&=0,\\ S\beta\phi\beta&=-1, & S\beta\phi\gamma&=0,\\ S\gamma\phi\gamma&=-1, & S\gamma\phi\alpha&=0;\end{aligned}$$

the six scalar conditions requisite (§ 178) for the determination of the self-conjugate linear and vector function $\phi$.

They give $$\alpha \parallel V\phi\beta\phi\gamma,$$
or $$x\alpha = \phi^{-1}V\beta\gamma.$$
Hence $$-x = xS\alpha\phi\alpha = S.\alpha\beta\gamma,$$
and similarly for the other combinations. Thus, as we have

$$\rho S.\alpha\beta\gamma = \alpha S.\beta\gamma\rho + \beta S.\gamma\alpha\rho + \gamma S.\alpha\beta\rho,$$

we find at once

$$-\phi\rho S^2.\alpha\beta\gamma = V\beta\gamma S.\beta\gamma\rho + V\gamma\alpha S.\gamma\alpha\rho + V\alpha\beta S.\alpha\beta\rho\,;$$

and the required equation may be put in the form

$$S^2.\alpha\beta\gamma = S^2.\alpha\beta\rho + S^2.\beta\gamma\rho + S^2.\gamma\alpha\rho.$$

The immediate interpretation is that *if four tetrahedra be formed by grouping, three and three, a set of semi-conjugate vector axes of an ellipsoid and any fourth vector of the surface, the sum of the squares of the volumes of three of these tetrahedra is equal to the square of the volume of the fourth.*

285. *A line moves with three of its points in given planes, find the locus of any fourth point.*

Let $a$, $b$, $c$ be the distances of the three points from the fourth, $\alpha$, $\beta$, $\gamma$ unit-vectors perpendicular to the planes respectively. If $\rho$ be the vector of the fourth point, referred to the point of intersection of the planes, and $\sigma$ a unit-vector parallel to the line, we have at once

$$S.\alpha(\rho - a\sigma) = 0,$$
$$S.\beta(\rho - b\sigma) = 0,$$
$$S.\gamma(\rho - c\sigma) = 0.$$

Thus

$$\sigma S.\alpha\beta\gamma = \frac{V\beta\gamma}{a}S\alpha\rho + \frac{V\gamma\alpha}{b}S\beta\rho + \frac{V\alpha\beta}{c}S\gamma\rho.$$

The condition $T\sigma = 1$ gives the equation of an ellipsoid referred to its centre.

We may write the equation in the form

$$\rho S\alpha\beta\gamma = aV\beta\gamma S\alpha\sigma + bV\gamma\alpha S\beta\sigma + cV\alpha\beta S\gamma\sigma$$
$$= \phi\sigma.S\alpha\beta\gamma, \text{ suppose,}$$

and from this we find at once for the volume of the ellipsoid

$$\pi\frac{S.\phi'\alpha\phi'\beta\phi'\gamma}{S.\alpha\beta\gamma} = \pi abc\,;$$

altogether independent of the relative inclinations of the three planes. This curious extension of a theorem of Monge is due to Booth.

286. We see from § 270 that (as in § 31 (*m*)) we can write the equation of an ellipsoid in the elegant form

$$\rho = \phi\epsilon,$$

where $\phi$ is a self-conjugate linear and vector function, and we impose the condition

$$T\epsilon = 1.$$

Hence, when the same ellipsoid is displaced by translation and rotation, by § 119 we may write its equation as

$$\rho = \delta + q\phi\epsilon q^{-1},$$

with the condition that $\epsilon$ is still a unit-vector.

Where it touches a plane perpendicular to $i$, we must have, simultaneously,

$$0 = S \,.\, iq\phi\epsilon' q^{-1} = S \,.\, q^{-1}iq\phi\epsilon'$$

and $$0 = S\epsilon\epsilon'.$$

Hence $$\epsilon = -U\phi\,(q^{-1}iq)$$

at the point of contact; and, if the plane touched be that of $jk$,

$$0 = Si\delta - S \,.\, q^{-1}iq\phi U\phi\,(q^{-1}iq),$$

or $$0 = Si\delta + T\phi\,(q^{-1}iq).$$

Thus, if we write

$$q^{-1}iq = \alpha, \quad q^{-1}jq = \beta, \quad q^{-1}kq = \gamma,$$

we have

$$\delta = +iT\phi\alpha + jT\phi\beta + kT\phi\gamma,$$

which gives the possible positions of the centre of a *given* ellipsoid when it is made to touch the fixed cöordinate planes.

We see by § 279 that $T\delta$ is constant. And it forms an interesting, though very simple, problem, to find the region of this spherical surface to which the position of the centre of the ellipsoid is confined. This, of course, involves giving to $\alpha$, $\beta$, $\gamma$ all possible values as a rectangular unit system.

287. For an investigation of the regions, on each of the cöordinate planes, within which the point of contact is confined, see a quaternion paper by Plarr (*Trans. R. S. E.* 1887). The difficulty of this question lies almost entirely in the eliminations, which are of a very formidable character. Subjoined is a mere sketch of one mode of solution, based on the preceding section.

The value of $\rho$, at the point of contact with

$$S i\rho = 0,$$

is

$$\delta - q\phi\,(U\phi\alpha)\,q^{-1},$$

or

$$+\Sigma i T\phi\alpha + \Sigma i S\,.\,q^{-1} i q\phi(U\phi\alpha),$$

or, finally,

$$j\left(T\phi\beta + \frac{S\,.\,\phi\alpha\phi\beta}{T\phi\alpha}\right) + k\left(T\phi\gamma + \frac{S\,.\,\phi\alpha\phi\gamma}{T\phi\alpha}\right).$$

Hence, in ordinary polar coördinates, the point of contact with the plane of $jk$ is

$$\left.\begin{aligned} r\cos\theta &= T\phi\beta + \frac{S\,.\,\phi\alpha\phi\beta}{T\phi\alpha} \\ r\sin\theta &= T\phi\gamma + \frac{S\,.\,\phi\alpha\phi\gamma}{T\phi\alpha} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

To find the boundary of the region within which the point of contact must lie, we must make $r$ a maximum or minimum, $\theta$ being constant, and $\alpha$, $\beta$, $\gamma$ being connected by the relations

$$\left.\begin{aligned} T\alpha\ &= T\beta\ = T\gamma\ = 1 \\ S\alpha\beta &= S\beta\gamma = S\gamma\alpha = 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Differentiating (1) and (2), with the conditions

$$d\theta = 0,$$

(because $\theta$ is taken as constant) and

$$dr = 0$$

as the criterion of the maximum, we have eight equations which are linear and homogeneous in $d\alpha$, $d\beta$, $d\gamma$. Eliminating the two latter among the seven equations which contain them, we have

$$0 = Sd\alpha\,[(\beta S\alpha\beta_1 + \beta_2)\,S\beta\gamma_1 + (\gamma S\alpha\gamma_1 + \gamma_2)\,S\gamma\beta_1]\ldots\ldots(3)$$

where

$$\beta_1 = \phi\,(U\phi\alpha - U\phi\beta),$$

$$\gamma_1 = \phi\,(U\phi\alpha - U\phi\gamma),$$

$$\beta_2 = -\frac{m}{T\phi\alpha^3}\,\phi V.\,\phi\alpha\phi^{-1}\gamma,$$

$$\gamma_2 = +\frac{m}{T\phi\alpha^3}\,\phi V.\,\phi\alpha\phi^{-1}\beta.$$

But we have also, as yet unemployed,

$$S\alpha d\alpha = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Conditions (3) and (4) give the two scalar equations

$$\left.\begin{aligned} 0 &= (S\beta_2\beta - S\beta_1\alpha)\,S\gamma_1\beta + S\gamma_2\beta\,S\beta_1\gamma \\ 0 &= S\beta_2\gamma\,S\gamma_1\beta + (S\gamma_2\gamma - S\gamma_1\alpha)\,S\beta_1\gamma \end{aligned}\right\}\ldots\ldots\ldots(5).$$

Theoretically, the ten equations (1), (2), and (5), suffice to eliminate the nine scalars involved in $\alpha, \beta, \gamma$ :—and thus to leave a single equation in $r, \theta$, which is that of the boundary of the region in question.

The student will find it a useful exercise to work out fully the steps required for the deduction of (5).

288. When the equation of a surface of the second order can be put in the form

$$S\rho\phi^{-1}\rho = -1 \quad \text{.........................(1)},$$

where

$$(\phi - g)(\phi - g_1)(\phi - g_2) = 0,$$

we know that $g, g_1, g_2$ are the squares of the principal semi-diameters. Hence, if we put $\phi + h$ for $\phi$ we have a second surface, the differences of the squares of whose principal semi-axes are the same as for the first. That is,

$$S\rho\,(\phi + h)^{-1}\rho = -1 \quad \text{.....................(2)},$$

is a surface *confocal* with (1). From this simple modification of the equation all the properties of a series of confocal surfaces may easily be deduced. We give a couple of examples.

*Any two confocal surfaces of the second degree, which meet, intersect at right angles.*

For the normal to (2) is evidently parallel to

$$(\phi + h)^{-1}\rho\,;$$

and that to another of the series, if it passes through the common point whose vector is $\rho$, is parallel to

$$(\phi + h_1)^{-1}\rho.$$

But

$$S\,.\,(\phi + h)^{-1}\rho\,(\phi + h_1)^{-1}\rho = S\,.\,\rho\,\frac{1}{(\phi + h)(\phi + h_1)}\,\rho$$

$$= \frac{1}{h - h_1}\,S\rho\,\{(\phi + h_1)^{-1} - (\phi + h)^{-1}\}\,\rho,$$

and this evidently vanishes if $h$ and $h_1$ are different, as they must be unless the surfaces are identical.

*To find the locus of the points of contact of a series of confocal surfaces with a series of parallel planes.*

Here the direction of the normal at the point, $\rho$, of contact is

$$(\phi + h)^{-1}\rho,$$

and is parallel to the common normal of the planes, say $\alpha$. Thus

$$(\phi + h)^{-1}\rho \parallel \alpha.$$

Thus the locus has the equation of a *plane* curve

$$\rho = x\alpha + y\phi\alpha,$$

and the relation between $x$ and $y$ is, by the general equation of the confocals,

$$1 + y^2 S\alpha\phi\alpha + xy\alpha^2 = 0.$$

Hence the locus is a hyperbola.

289. *To find the conditions of similarity of two central surfaces of the second degree.*

Referring them to their centres, let their equations be

$$\left.\begin{array}{l} S\rho\phi\rho = -1 \\ S\rho\psi\rho = -1 \end{array}\right\} \text{.........................(1).}$$

Now the obvious conditions are that the axes of the one are proportional to those of the other. Hence, if

$$\left.\begin{array}{l} g^3 - m_2 g^2 + m_1 g - m = 0 \\ g'^3 - m'_2 g'^2 + m'_1 g' - m' = 0 \end{array}\right\} \text{...............(2),}$$

be the equations for determining the squares of the reciprocals of the semi-axes, we must have

$$\frac{m'_2}{m_2} = \mu, \quad \frac{m'_1}{m_1} = \mu^2, \quad \frac{m'}{m} = \mu^3 \text{..................(3),}$$

where $\mu$ is an undetermined scalar. Thus it appears that there are but two scalar conditions necessary. Eliminating $\mu$ we have

$$\frac{m'^2_2}{m_2^2} = \frac{m'_1}{m_1}, \quad \frac{m'm'_2}{mm_2} = \frac{m'^2_1}{m_1^2} \text{..................(4),}$$

which are equivalent to the ordinary conditions.

290. *Find the greatest and least semi-diameters of a central plane section of an ellipsoid.*

Here

$$\left.\begin{array}{r} S\rho\phi\rho = -1 \\ S\alpha\rho = 0 \end{array}\right\} \text{...........................(1)}$$

together represent the elliptic section; and our additional condition is that $T\rho$ is a maximum or minimum.

Differentiating the equations of the ellipse, we have

$$S\phi\rho d\rho = 0,$$

$$S\alpha d\rho = 0,$$

and the maximum condition gives

$$dT\rho = 0,$$

or

$$S\rho d\rho = 0.$$

Eliminating the indeterminate vector $d\rho$ we have

$$S.\alpha\rho\phi\rho = 0 \text{ ..........................(2).}$$

This shews that *the maximum or minimum vector, the normal at its extremity, and the perpendicular to the plane of section, lie in one plane.* It also shews that there are but two vector-directions which satisfy the conditions, and that they are perpendicular to each other, for (2) is unchanged if $\alpha\rho$ be substituted for $\rho$.

We have now to solve the three equations (1) and (2), to find the vectors of the two (four) points in which the ellipse (1) intersects the cone (2). We obtain at once

$$\phi\rho = xV.(\phi^{-1}\alpha)\,V\alpha\rho.$$

Operating by $S.\rho$ we have

$$-1 = x\rho^2 S\alpha\phi^{-1}\alpha.$$

Hence
$$-\rho^2\phi\rho = \rho - \alpha\frac{S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha}$$

or
$$\rho = \frac{S\rho\phi^{-1}\alpha}{S\alpha\phi^{-1}\alpha}(1+\rho^2\phi)^{-1}\alpha \text{ ...............(3);}$$

from which
$$S.\alpha(1+\rho^2\phi)^{-1}\alpha = 0 \text{ .....................(4);}$$

a quadratic equation in $\rho^2$, from which the lengths of the maximum and minimum vectors are to be determined. By § 184 it may be written

$$m\rho^4 S\alpha\phi^{-1}\alpha + \rho^2 S.\alpha(m_2-\phi)\alpha + \alpha^2 = 0 \text{ .........(5).}$$

[If we had operated by $S.\phi^{-1}\alpha$ or by $S.\phi^{-1}\rho$, instead of by $S.\rho$, we should have obtained an equation apparently different from this, but easily reducible to it. To prove their identity is a good exercise for the student.]

Substituting the values of $\rho^2$ given by (5) in (3) we obtain the vectors of the required diameters. [The student may easily prove directly that

$$(1+\rho_1^{\,2}\phi)^{-1}\alpha \quad\text{and}\quad (1+\rho_2^{\,2}\phi)^{-1}\alpha$$

are necessarily perpendicular to each other, if both be perpendicular to $\alpha$, and if $\rho_1^{\,2}$ and $\rho_2^{\,2}$ be different. See § 288.]

291. By (5) of last section we see that

$$\rho_1^{\,2}\rho_2^{\,2} = \frac{\alpha^2}{mS\alpha\phi^{-1}\alpha}.$$

Hence the area of the ellipse (1) is

$$\frac{\pi T\alpha}{\sqrt{-mS\alpha\phi^{-1}\alpha}}.$$

Also the locus of central normals to all diametral sections of an ellipsoid, whose areas are equal, is the cone

$$S\alpha\phi^{-1}\alpha = C\alpha^2.$$

When the roots of (5) are equal, i.e. when

$$(m_2\alpha^2 - S\alpha\phi\alpha)^2 = 4m\alpha^2 S\alpha\phi^{-1}\alpha \quad \ldots\ldots\ldots\ldots\ldots(6),$$

the section is a circle. It is not difficult to prove that this equation is satisfied by only two values of $U\alpha$, but another quaternion form of the equation gives the solution of this and similar problems by inspection. (See § 292 below.)

292. By § 180 we may write the equation

$$S\rho\phi\rho = -1$$

in the new form $\qquad S.\lambda\rho\mu\rho + p\rho^2 = -1,$

where $p$ is a known scalar, and $\lambda$ and $\mu$ are definitely known (with the exception of their tensors, whose product alone is given) in terms of the constants involved in $\phi$. [The reader is referred again also to §§ 128, 129.] This may be written

$$2S\lambda\rho S\mu\rho + (p - S\lambda\mu)\rho^2 = -1 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

From this form it is obvious that the surface is cut by any plane perpendicular to $\lambda$ or $\mu$ in a circle. For, if we put

$$S\lambda\rho = a,$$

we have $\qquad 2aS\mu\rho + (p - S\lambda\mu)\rho^2 = -1,$

the equation of a sphere which passes through the plane curve of intersection.

Hence $\lambda$ and $\mu$ of § 180 are the values of $\alpha$ in equation (6) of the preceding section.

293. *Any two circular sections of a central surface of the second degree, whose planes are not parallel, lie on a sphere.*

For the equation

$$(S\lambda\rho - a)(S\mu\rho - b) = 0,$$

where $a$ and $b$ are any scalar constants whatever, is that of a system of two non-parallel planes, cutting the surface in circles. Eliminating the product $S\lambda\rho S\mu\rho$ between this and equation (1) of last section, there remains the equation of a sphere.

294. *To find the generating lines of a central surface of the second degree.*

Let the equation be

$$S\rho\phi\rho = -1;$$

then, if $\alpha$ be the vector of any point on the surface, and $\varpi$ a vector parallel to a generating line, we must have

$$\rho = \alpha + x\varpi$$

for all values of the scalar $x$.

Hence $$S(\alpha + x\varpi)\,\phi\,(\alpha + x\varpi) = -1$$

gives the two equations

$$\left.\begin{aligned} S\alpha\phi\varpi &= 0 \\ S\varpi\phi\varpi &= 0 \end{aligned}\right\}.$$

The first is the equation of a plane through the origin parallel to the tangent plane at the extremity of $\alpha$, the second is the equation of the asymptotic cone. The generating lines are therefore parallel to the intersections of these two surfaces, as is well known.

From these equations we have

$$y\phi\varpi = V\alpha\varpi$$

where $y$ is a scalar to be determined. Operating on this by $S.\beta$ and $S.\gamma$, where $\beta$ and $\gamma$ are any two vectors not coplanar with $\alpha$, we have

$$S\varpi\,(y\phi\beta + V\alpha\beta) = 0, \quad S\varpi\,(y\phi\gamma - V\gamma\alpha) = 0 \quad \ldots\ldots(1).$$

Hence $$S.\phi\alpha\,(y\phi\beta + V\alpha\beta)\,(y\phi\gamma - V\gamma\alpha) = 0,$$

or $$my^2 S.\alpha\beta\gamma - S\alpha\phi\alpha S.\alpha\beta\gamma = 0.$$

Thus we have the two values

$$y = \pm\sqrt{\frac{S\alpha\phi\alpha}{m}} = \pm\sqrt{-\frac{1}{m}},$$

belonging to the two generating lines. That they may be real it is clear that $m$ must be negative:—i.e. the surface must be the one-sheeted hyperboloid.

295. But by equations (1) we have

$$\begin{aligned} z\varpi &= V.(y\phi\beta + V\alpha\beta)(y\phi\gamma - V\gamma\alpha) \\ &= my^2\phi^{-1}V\beta\gamma + yV.\phi\alpha V\beta\gamma - \alpha S.\alpha V\beta\gamma; \end{aligned}$$

which, according to the sign of $y$, gives one or other generating line.

Here $V\beta\gamma$ may be any vector whatever, provided it is not perpendicular to $\alpha$ (a condition assumed in last section), and we may write for it $\theta$.

Substituting the value of $y$ before found, we have

$$z\varpi = -\phi^{-1}\theta - \alpha S\alpha\theta \pm \sqrt{-\frac{1}{m}}.V\theta\phi\alpha,$$

$$=V.\,\phi\alpha V\alpha\phi^{-1}\theta \pm \sqrt{-\frac{1}{m}}.\,V\theta\phi\alpha,$$

or, as we may evidently write it,

$$=\phi^{-1}(V.\,\alpha V\phi\alpha\theta) \pm \sqrt{-\frac{1}{m}}.\,V\theta\phi\alpha \ldots\ldots\ldots\ldots (2).$$

Put $\tau = V\theta\phi\alpha,$

and we have $z\varpi = -\,\phi^{-1}V\alpha\tau \pm \sqrt{-\frac{1}{m}}\,.\,\tau,$

with the condition $S\tau\phi\alpha = 0.$

[Any one of these sets of values forms the complete solution of the problem; but more than one have been given, on account of their singular nature and the many properties of surfaces of the second degree which immediately follow from them. It will be excellent practice for the student to shew that

$$\psi\theta = U\left(V.\,\phi\alpha V\alpha\phi^{-1}\theta \pm \sqrt{-\frac{1}{m}}.V\theta\phi\alpha\right)$$

is an invariant. This may most easily be done by proving that

$$V.\,\psi\theta\psi\theta_1 = 0 \quad \text{identically.]}$$

Perhaps, however, it is simpler to write $\alpha$ for $V\beta\gamma$, and we thus obtain

$$z\varpi = -\,\phi^{-1}V.\,\alpha V\alpha\phi\alpha \mp \sqrt{-\frac{1}{m}}.V\alpha\phi\alpha.$$

[The reader need hardly be reminded that we are dealing with the *general* equation of the central surfaces of the second degree—the centre being origin.]

## EXAMPLES TO CHAPTER IX.

1. Find the locus of points on the surface

$$S\rho\phi\rho = -1$$

where the generating lines are at right angles to one another.

2. Find the equation of the surface described by a straight line which revolves about an axis, which it does not meet, but with which it is rigidly connected.

3. Find the conditions that

$$S\rho\phi\rho = -1$$

may be a surface of revolution, with axis parallel to a given vector.

4. Find the equations of the right cylinders which circumscribe a given ellipsoid.

5. Find the equation of the locus of the extremities of perpendiculars to central plane sections of an ellipsoid, erected at the centre, their lengths being the principal semi-axes of the sections. [Fresnel's *Wave-Surface*. See Chap. XII.]

6. The cone touching central plane sections of an ellipsoid, which are of equal area, is asymptotic to a confocal hyperboloid.

7. Find the envelop of all non-central plane sections of an ellipsoid when their area is constant.

8. Find the locus of the intersection of three planes, perpendicular to each other, and touching, respectively, each of three confocal surfaces of the second degree.

9. Find the locus of the foot of the perpendicular from the centre of an ellipsoid upon the plane passing through the extremities of a set of conjugate diameters.

10. Find the points in an ellipsoid where the inclination of the normal to the radius-vector is greatest.

11. If four similar and similarly situated surfaces of the second degree intersect, the planes of intersection of each pair pass through a common point.

12. If a parallelepiped be inscribed in a central surface of the second degree its edges are parallel to a system of conjugate diameters.

13. Shew that there is an infinite number of sets of axes for which the Cartesian equation of an ellipsoid becomes

$$x^2 + y^2 + z^2 = e^2.$$

14. Find the equation of the surface of the second degree which circumscribes a given tetrahedron so that the tangent plane at each angular point is parallel to the opposite face; and shew that its centre is the mean point of the tetrahedron.

15. Two similar and similarly situated surfaces of the second degree intersect in a plane curve, whose plane is conjugate to the vector joining their centres.

16. Find the locus of all points on

$$S\rho\phi\rho = -1,$$

where the normals meet the normal at a given point.

Also the locus of points on the surface, the normals at which meet a given line in space.

17. Normals drawn at points situated on a generating line are parallel to a fixed plane.

18. Find the envelop of the planes of contact of tangent planes drawn to an ellipsoid from points of a concentric sphere. Find the locus of the point from which the tangent planes are drawn if the envelop of the planes of contact is a sphere.

19. The sum of the reciprocals of the squares of the perpendiculars from the centre upon three conjugate tangent planes is constant.

20. Cones are drawn, touching an ellipsoid, from any two points of a similar, similarly situated, and concentric ellipsoid. Shew that they intersect in two plane curves.

Find the locus of the vertices of the cones when these plane sections are at right angles to one another.

21. Any two tangent cylinders to an ellipsoid intersect in two plane ellipses, and no other tangent cylinder can be drawn through either of these.

Find the locus of these ellipses:—

(*a*) When the axes of the two cylinders are conjugate to each other, and to a *given* diameter.

(*b*) When they are conjugate to each other, and to diameters lying in one plane.

(*c*) When they are conjugate to each other, and to any diameter whatever.

22. If $\alpha$, $\beta$, $\gamma$ be unit vectors parallel to conjugate semi-diameters of an ellipsoid, what is the vector

$$\sigma = \frac{\alpha}{\sqrt{-S\alpha\phi\alpha}} + \frac{\beta}{\sqrt{-S\beta\phi\beta}} + \frac{\gamma}{\sqrt{-S\gamma\phi\gamma}},$$

and what the locus of its extremity?

23. Find the locus of the points of contact of tangent planes which are equidistant from the centre of a surface of the second degree.

24. From a fixed point $A$, on the surface of a given sphere, draw any chord $AD$; let $D'$ be the second point of intersection of the sphere with the secant $BD$ drawn from any point $B$; and take a radius vector $AE$, equal in length to $BD'$, and in direction either coincident with, or opposite to, the chord $AD$: the locus of $E$ is an ellipsoid, whose centre is $A$, and which passes through $B$. (Hamilton, *Elements*, p. 227.)

25. Shew that the equation

$$l^2(e^2-1)(e+S\alpha\alpha') = (S\alpha\rho)^2 - 2e\,S\alpha\rho\,S\alpha'\rho + (S\alpha'\rho)^2 + (1-e^2)\rho^2,$$

where $e$ is a variable (scalar) parameter, and $\alpha$, $\alpha'$ unit-vectors, represents a system of confocal surfaces. (*Ibid.* p. 644.)

26. Shew that the locus of the diameters of

$$S\rho\phi\rho = -1,$$

which are parallel to the chords bisected by the tangent planes to the cone

$$S\rho\psi\rho = 0,$$

is the cone

$$S.\rho\phi\psi^{-1}\phi\rho = 0.$$

27. Find the equation of a cone, whose vertex is one summit of a given tetrahedron, and which passes through the circle circumscribing the opposite side.

28. Shew that the locus of points on the surface

$$S\rho\phi\rho = -1,$$

the normals at which meet that drawn at the point $\rho = \varpi$, is on the cone

$$S.(\rho-\varpi)\,\phi\varpi\phi\rho = 0.$$

29. Find the equation of the locus of a point the square of whose distance from a given line is proportional to its distance from a given plane.

30. Shew that the locus of the pole of the plane

$$S\alpha\rho = 1,$$

with respect to the surface

$$S\rho\phi\rho = -1,$$

is a sphere, if $\alpha$ be subject to the condition

$$S\alpha\phi^{-2}\alpha = C.$$

31. Shew that the equation of the surface generated by lines drawn through the origin parallel to the normals to

$$S\rho\phi^{-1}\rho = -1$$

along its lines of intersection with

$$S\rho\,(\phi + h)^{-1}\rho = -1,$$

is

$$\varpi^2 - hS\varpi\,(\phi + h)^{-1}\varpi = 0.$$

32. Common tangent planes are drawn to

$$2S\lambda\rho S\mu\rho + (p - S\lambda\mu)\,\rho^2 = -1, \quad \text{and} \quad T\rho = h,$$

find the value of $h$ that the lines of contact with the former surface may be plane curves. What are they, in this case, on the sphere?

Discuss the case of

$$p^2 - S^2\lambda\mu = 0.$$

33. If tangent cones be drawn to

$$S\rho\phi^2\rho = -1,$$

from every point of

$$S\rho\phi\rho = -1,$$

the envelop of their planes of contact is

$$S\rho\phi^3\rho = -1.$$

34. Tangent cones are drawn from every point of

$$S\,(\rho - \alpha)\,\phi\,(\rho - \alpha) = -n^2,$$

to the similar and similarly situated surface

$$S\rho\phi\rho = -1,$$

shew that their planes of contact envelop the surface

$$(S\alpha\phi\rho + 1)^2 = -n^2\,S\rho\phi\rho.$$

35. Find the envelop of planes which touch the parabolas

$$\rho = \alpha t^2 + \beta t, \qquad \rho = \alpha u^2 + \gamma u,$$

where $\alpha$, $\beta$, $\gamma$ form a rectangular system, and $t$ and $u$ are scalars.

36. Find the equation of the surface on which lie the lines of contact of tangent cones drawn from a fixed point to a series of similar, similarly situated, and concentric ellipsoids.

37. Discuss the surfaces whose equations are

$$S\alpha\rho S\beta\rho = S\gamma\rho,$$

and

$$S^2\alpha\rho + S\,.\,\alpha\beta\rho = 1.$$

38. Shew that the locus of the vertices of the right cones which touch an ellipsoid is a hyperbola.

39. If $\alpha_1$, $\alpha_2$, $\alpha_3$ be vector conjugate diameters of

$$S\rho\phi\rho = -1,$$

where
$$\phi^3 - m_2\phi^2 + m_1\phi - m = 0,$$
shew that

$$\Sigma(\alpha^2) = -\frac{m_1}{m}, \quad \Sigma(V\alpha_1\alpha_2)^2 = -\frac{m_2}{m}, \quad S^2.\alpha_1\alpha_2\alpha_3 = \frac{1}{m},$$

and
$$\Sigma(\phi\alpha)^2 = -m_2.$$

40. Find the locus of the lines of contact of tangent planes from a given point to a series of spheres, whose centres are in one line and which pass through a given point in that line.

41. Find the locus of a circle of variable radius, whose plane is always parallel to a given plane, and which passes through each of three given lines in space.

# CHAPTER X.

## GEOMETRY OF CURVES AND SURFACES.

296. We have already seen (§ 31 ($l$)) that the equations

$$\rho = \phi t = \Sigma . \alpha f(t),$$

and

$$\rho = \phi (t, u) = \Sigma . \alpha f(t, u),$$

where $\alpha$ represents one of a set of given vectors, and $f$ a scalar function of scalars $t$ and $u$, represent respectively a curve and a surface. We commence the present too brief Chapter with a few of the immediate deductions from these forms of expression. We shall then give a number of examples, with little attempt at systematic development or even arrangement.

297. What may be denoted by $t$ and $u$ in these equations is, of course, quite immaterial: but in the case of curves, considered geometrically, $t$ is most conveniently taken as the length, $s$, of the curve, measured from some fixed point. In the Kinematical investigations of the next Chapter $t$ may, with great convenience, be employed to denote *time.*

298. Thus we may write the equation of any curve in space as

$$\rho = \phi s,$$

where $\phi$ is a vector function of the length, $s$, of the curve. Of course it is a *linear* function when, and only when, the equation (as in § 31 ($l$)) represents a straight line.

If $\phi$ be a periodic function, such that

$$\phi (l + s) = \phi (s),$$

the curve is a rëentrant one, generally a *Knot* in space.

[In § 306 it is shewn that when $s$ is the arc certain forms of $\phi$ are not admissible.]

299. We have also seen (§§ 38, 39) that

$$\frac{d\rho}{ds} = \frac{d}{ds}\phi s = \phi' s$$

is a vector of *unit* length in the direction of the tangent at the extremity of $\rho$.

At the proximate point, denoted by $s + \delta s$, this unit tangent vector becomes

$$\phi' s + \phi'' s \delta s + \&c.$$

But, because $$T\phi' s = 1,$$

we have $$S . \phi' s \phi'' s = 0.$$

Hence $\phi'' s$, which is a vector in the osculating plane of the curve, is also perpendicular to the tangent.

Also, if $\delta\theta$ be the angle between the successive tangents $\phi' s$ and $\phi' s + \phi'' s \delta s + \ldots\ldots$, we have

$$\mathcal{L} \frac{\delta\theta}{\delta s} = T\phi'' s;$$

so that *the tensor of $\phi'' s$ is the reciprocal of the radius of absolute curvature at the point s.*

300. Thus, if $\overline{OP} = \phi s$ be the vector of any point $P$ of the curve, and if $C$ be the centre of curvature at $P$, we have

$$\overline{PC} = -\frac{1}{\phi'' s};$$

and thus $$\overline{OC} = \phi s - \frac{1}{\phi'' s}$$

is the equation of the locus of the centre of curvature.

Hence also $$V . \phi' s \phi'' s \text{ or } \phi' s \phi'' s$$

is a vector perpendicular to the osculating plane; and therefore

$$T\frac{d}{ds}(\phi' s U \phi'' s)$$

is the *tortuosity* of the given curve, or the rate of rotation of its osculating plane per unit of length.

301. As an example of the use of these expressions let us *find the curve whose curvature and tortuosity are both constant.*

We have $$\text{curvature} = T\phi'' s = T\rho'' = c.$$

Hence $$\phi' s \phi'' s = \rho' \rho'' = c\alpha,$$
where $\alpha$ is a unit vector perpendicular to the osculating plane. This gives
$$\rho' \rho''' + \rho''^2 = c \mathcal{L} \frac{\delta\alpha}{\delta s} = cc_1 U\rho'' = c_1 \rho'',$$
if $c_1$ represent the tortuosity.

Integrating we get
$$\rho' \rho'' = c_1 \rho' + \beta \text{ ..........................(1)},$$
where $\beta$ is a constant vector. Squaring both sides of this equation, we get
$$c^2 = c_1^2 - \beta^2 - 2c_1 S\beta\rho'$$
$$= - c_1^2 - \beta^2$$
(for by operating with $S . \rho'$ upon (1) we get $+ c_1 = S\beta\rho'$),

or $$T\beta = \sqrt{c^2 + c_1^2}.$$
Multiply (1) by $\rho'$, remembering that
$$To = 1,$$
and we obtain $$- \rho'' = - c_1 + \rho' \beta,$$
or, by integration, $$\rho' = c_1 s - \rho\beta + a \text{ ......................(2)},$$
where $a$ is a constant quaternion. Eliminating $\rho'$, we have
$$- \rho'' = - c_1 + c_1 s\beta - \rho\beta^2 + a\beta,$$
of which the vector part is
$$\rho'' - \rho\beta^2 = - c_1 s\beta - Va\beta.$$
The complete integral of this equation is evidently
$$\rho = \xi \cos . sT\beta + \eta \sin . sT\beta - \frac{1}{T\beta^2}(c_1 s\beta + Va\beta) \text{ ......(3)},$$
$\xi$ and $\eta$ being any two constant vectors. We have also by (2),
$$S\beta\rho = c_1 s + Sa,$$
which requires that $S\beta\xi = 0, \quad S\beta\eta = 0.$

The farther test, that $T\rho' = 1$, gives us
$$-1 = T\beta^2(\xi^2 \sin^2 . sT\beta + \eta^2 \cos^2 . sT\beta - 2S\xi\eta \sin . sT\beta \cos . sT\beta) - \frac{c_1^2}{c^2 + c_1^2}.$$
This requires, of course,
$$S\xi\eta = 0, \qquad T\xi = T\eta = \frac{c}{c^2 + c_1^2},$$
so that (3) becomes the general equation of a helix traced on a right cylinder. (Compare § 31 (*m*).)

302. The vector perpendicular from the origin on the tangent to the curve

$$\rho = \phi s$$

is, of course, $\frac{1}{\rho'} V\rho'\rho$, or $\rho' V\rho\rho'$

(since $\rho'$ is a unit vector).

*To find a common property of curves whose tangents are all equidistant from the origin.*

Here

$$TV\rho\rho' = c,$$

which may be written $-\rho^2 - S^2\rho\rho' = c^2$ ...................... (1).

This equation shews that, as is otherwise evident, *every curve on a sphere whose centre is the origin* satisfies the condition. For obviously

$$-\rho^2 = c^2 \text{ gives } S\rho\rho' = 0,$$

and these satisfy (1).

If $S\rho\rho'$ does not vanish, the integral of (1) is

$$\sqrt{T\rho^2 - c^2} = s \text{..........................(2)},$$

an arbitrary constant not being necessary, as we may measure $s$ from any point of the curve. The equation of an involute which commences at this assumed point is

$$\varpi = \rho - s\rho'.$$

This gives
$$\begin{aligned} T\varpi^2 &= T\rho^2 + s^2 + 2sS\rho\rho' \\ &= T\rho^2 + s^2 - 2s\sqrt{T\rho^2 - c^2}, \quad \text{by (1)}, \\ &= c^2, \quad \text{by (2)}. \end{aligned}$$

This includes *all curves whose involutes lie on a sphere about the origin.*

303. *Find the locus of the foot of the perpendicular drawn to a tangent to a right helix from a point in the axis.*

The equation of the helix is

$$\rho = \alpha \cos \frac{s}{a} + \beta \sin \frac{s}{a} + \gamma s,$$

where the vectors $\alpha$, $\beta$, $\gamma$ are at right angles to each other, and

$$T\alpha = T\beta = b, \quad \text{while} \quad aT\gamma = \sqrt{a^2 - b^2}.$$

(The latter condition is from $T\rho' = 1$.)

The equation of the required locus is, by last section,

$$\varpi = \rho' V\rho\rho'$$

$$= \alpha\left(\cos\frac{s}{a} + \frac{a^2-b^2}{a^3}s\sin\frac{s}{a}\right) + \beta\left(\sin\frac{s}{a} - \frac{a^2-b^2}{a^3}s\cos\frac{s}{a}\right) + \gamma\frac{b^2}{a^2}s.$$

This curve lies on the hyperboloid whose equation is

$$S^2\alpha\varpi + S^2\beta\varpi - a^2S^2\gamma\varpi = b^4,$$

as the reader may easily prove for himself.

304. *To find the least distance between consecutive tangents to a tortuous curve.*

Let one tangent be $\varpi = \rho + x\rho'$,

then a consecutive one, at a distance $\delta s$ along the curve, is

$$\varpi = \rho + \rho'\delta s + \rho''\frac{\delta s^2}{1\,.\,2} + \&c. + y\left(\rho' + \rho''\delta s + \rho'''\frac{\delta s^2}{1\,.\,2} + \ldots\right).$$

The magnitude of the least distance between these lines is, by §§ 216, 223,

$$S.\left(\rho'\delta s + \rho''\frac{\delta s^2}{1.2} + \rho'''\frac{\delta s^3}{1.2.3} + \ldots\right)UV.\rho'\left(\rho' + \rho''\delta s + \rho'''\frac{\delta s^2}{1.2} + \ldots\right)$$

$$= \frac{-\frac{\delta s^4}{12}S\,.\,\rho'\rho''\rho'''}{TV\rho'\rho''\delta s},$$

if we neglect terms of higher orders.

It may be written, since $\rho'\rho''$ is a vector, and $T\rho' = 1$,

$$\frac{\delta s^3}{12}S\,.\,U\rho''V\rho'\rho'''.$$

But (§ 140 (2)) $\dfrac{\delta UV\rho'\rho''}{UV\rho'\rho''} = V\dfrac{V\rho'\rho'''}{V\rho'\rho''}\delta s = \dfrac{\delta s}{\rho''^2}\rho'S\,.\,\rho'\rho''\rho'''.$

Hence $$\frac{\delta s}{T\rho''}S\,.\,U\rho''V\rho'\rho'''$$

is the small angle, $\delta\phi$, between the two successive positions of the osculating plane. [See also § 300.]

Thus the shortest distance between two consecutive tangents is expressed by the formula

$$\frac{\delta\phi\delta s^2}{12r},$$

where $r, = \dfrac{1}{T\rho''}$, is the radius of absolute curvature of the tortuous curve.

305. Let us recur for a moment to the equation of the parabola (§ 31 (*f*))

$$\rho = \alpha t + \frac{\beta t^2}{2}.$$

Here $$\rho' = (\alpha + \beta t)\frac{dt}{ds},$$

whence, if we assume $S\alpha\beta = 0$,

$$\frac{ds}{dt} = \sqrt{-\alpha^2 - \beta^2 t^2},$$

from which the length of the arc of the curve can be derived in terms of $t$ by integration.

Again, $$\rho'' = (\alpha + \beta t)\frac{d^2t}{ds^2} + \beta\left(\frac{dt}{ds}\right)^2.$$

But $$\frac{d^2t}{ds^2} = \frac{d}{ds}\cdot\frac{1}{T(\alpha+\beta t)} = +\frac{dt}{ds}\frac{S.\beta(\alpha+\beta t)}{T(\alpha+\beta t)^3}.$$

Hence $$\rho'' = -\frac{(\alpha+\beta t)\,V\alpha\beta}{T(\alpha+\beta t)^4},$$

and therefore, for the vector of the centre of curvature we have (§ 300),

$$\varpi = \alpha t + \frac{\beta t^2}{2} - (\alpha^2 + \beta^2 t^2)^2(-\beta\alpha^2 + \alpha\beta^2 t)^{-1},$$

$$= \beta\left(\frac{3t^2}{2} + \frac{\alpha^2}{\beta^2}\right) - \alpha\frac{t^3\beta^2}{\alpha^2};$$

which is the quaternion equation of the evolute.

306. One of the simplest forms of the equation of a tortuous curve is

$$\rho = \alpha t + \frac{\beta t^2}{2} + \frac{\gamma t^3}{6},$$

where $\alpha, \beta, \gamma$ are any three non-coplanar vectors, and the numerical factors are introduced for convenience. This curve lies on a parabolic cylinder whose generating lines are parallel to $\gamma$; and also on cylinders whose bases are a cubical and a semi-cubical parabola, their generating lines being parallel to $\beta$ and $\alpha$ respectively. We have by the equation of the curve

$$\rho' = \left(\alpha + \beta t + \frac{\gamma t^2}{2}\right)\frac{dt}{ds},$$

from which, by $T\rho' = 1$, the length of the curve can be found in terms of $t$; and

$$\rho'' = \left(\alpha + \beta t + \frac{\gamma t^2}{2}\right)\frac{d^2t}{ds^2} + (\beta + \gamma t)\left(\frac{dt}{ds}\right)^2,$$

from which $\rho''$ can be expressed in terms of $s$. The investigation of various properties of this curve is very easy, and will be of great use to the student.

[*Note.*—It is to be observed that in this equation $t$ cannot stand for $s$, the length of the curve. It is a good exercise for the student to shew that such an equation as

$$\rho = \alpha s + \beta s^2 + \gamma s^3,$$

or even the simpler form

$$\rho = \alpha s + \beta s^2,$$

involves an absurdity.]

307. The equation $\rho = \phi^t \epsilon$,

where $\phi$ is a given self-conjugate linear and vector function, $t$ a scalar variable, and $\epsilon$ an arbitrary vector constant, belongs to a curious class of curves.

We have at once $\frac{d\rho}{dt} = \phi^t \log \phi\, \epsilon$,

where $\log \phi$ is another self-conjugate linear and vector function, which we may denote by $\chi$. These functions are obviously commutative, as they have the same principal set of rectangular vectors, hence we may write

$$\frac{d\rho}{dt} = \chi\rho,$$

which of course gives $\frac{d^2\rho}{dt^2} = \chi^2\rho$, &c.,

since $\chi$ does not involve $t$.

As a verification, we should have

$$\phi^{t+\delta t}\,\epsilon = \rho + \frac{d\rho}{dt}\delta t + \frac{d^2\rho}{dt^2}\frac{\delta t^2}{1\,.\,2} + \&\text{c.}$$

$$= \left(1 + \delta t\chi + \frac{\delta t^2}{1\,.\,2}\chi^2 + \ldots\ldots\right)\rho$$

$$= e^{\delta t\chi}\,\rho,$$

where $e$ is the base of Napier's Logarithms.

This is obviously true if $\phi^{\delta t} = e^{\delta t\chi}$,

or $$\phi = e^{\chi},$$

or $$\log \phi = \chi,$$

which is our assumption. See § 337, below.

[The above process is, at first sight, rather startling, but the student may easily verify it by writing, in accordance with the results of Chapter V,

$$\phi\epsilon = - g_1 \alpha S\alpha\epsilon - g_2 \beta S\beta\epsilon - g_3 \gamma S\gamma\epsilon,$$

whence
$$\phi^t\epsilon = - g^t{}_1 \alpha S\alpha\epsilon - g^t{}_2 \beta S\beta\epsilon - g^t{}_3 \gamma S\gamma\epsilon.$$

He will find at once

$$\chi\epsilon = - \log g_1 \alpha S\alpha\epsilon - \log g_2 \beta S\beta\epsilon - \log g_3 \gamma S\gamma\epsilon,$$

and the results just given follow immediately.]

308. That the equation

$$\rho = \phi(t, u) = \Sigma . \alpha f(t, u)$$

represents a surface is obvious from the fact that it becomes the equation of a definite curve whenever *either* $t$ or $u$ has a particular value assigned to it. Hence the equation at once furnishes us with two systems of curves, lying wholly on the surface, and such that one of each system can, in general, be drawn through any assigned point on the surface. Tangents drawn to these curves at their point of intersection must, of course, lie in the tangent plane, whose equation we have thus the means of forming. [Of course, there may occasionally be cases of indeterminateness, as when the curves happen to *touch* one another. But the general consideration of singular points on surfaces is beyond the scope of this work.]

309. By the equation we have

$$d\rho = \left(\frac{d\phi}{dt}\right) dt + \left(\frac{d\phi}{du}\right) du,$$

where the brackets are inserted to indicate partial differential coefficients. If we write this as

$$d\rho = \phi'_t dt + \phi'_u du,$$

the normal to the tangent plane is evidently

$$V\phi'_t \phi'_u,$$

and the equation of that plane

$$S . (\varpi - \phi) \phi'_t \phi'_u = 0.$$

310. Thus, as a simple example, let

$$\rho = t\alpha + u\beta + tu\gamma.$$

This surface is evidently to be constructed by drawing through each point $t\alpha$, of the line $\alpha$, a line parallel to $\beta + t\gamma$; or through $u\beta$, a line parallel to $\alpha + u\gamma$.

We may easily eliminate $t$ and $u$, and obtain

$$S.\beta\gamma\rho S.\gamma\alpha\rho = S.\alpha\beta\gamma S.\alpha\beta\rho\,;$$

and the methods of last chapter enable us to recognise a hyperbolic paraboloid.

Again, suppose a straight line to move along a fixed straight line, remaining always perpendicular to it, while rotating about it through an angle proportional to the space it has advanced; the equation of the ruled surface described will evidently be

$$\rho = \alpha t + u\,(\beta\cos t + \gamma\sin t) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\alpha$, $\beta$, $\gamma$ are rectangular vectors, and

$$T\beta = T\gamma.$$

This surface evidently intersects the right cylinder

$$\rho = a\,(\beta\cos t + \gamma\sin t) + v\alpha,$$

in a helix (§§ 31 ($m$), 301) whose equation is

$$\rho = \alpha t + a\,(\beta\cos t + \gamma\sin t).$$

These equations illustrate very well the remarks made in §§ 31 ($l$), 308, as to the curves or surfaces represented by a vector equation according as it contains one or two scalar variables.

From (1) we have

$$d\rho = [\alpha - u\,(\beta\sin t - \gamma\cos t)]\,dt + (\beta\cos t + \gamma\sin t)\,du,$$

so that the normal at the extremity of $\rho$ is

$$T\alpha\,(\gamma\cos t - \beta\sin t) - u\,T\beta^2 U\alpha.$$

Hence, as we proceed along a generating line of the surface, for which $t$ is constant, we see that the direction of the normal changes. This, of course, proves that the surface is not developable.

311. Hence the criterion for a developable surface is that if it be expressed by an equation of the form

$$\rho = \phi t + u\psi t,$$

where $\phi t$ and $\psi t$ are vector functions, we must have the *direction* of the normal

$$V\{\phi' t + u\psi' t\}\,\psi t$$

independent of $u$.

This requires either $\quad V\psi t\psi' t = 0,$

which would reduce the surface to a cylinder, all the generating lines being parallel to each other; or

$$V\phi' t\psi t = 0.$$

This is the criterion we seek, and it shews that we may write, for a developable surface in general, the equation

$$\rho = \phi t + u\phi' t \quad\text{..........................}(1).$$

Evidently $\rho = \phi t$

is a curve (generally tortuous) and $\phi' t$ is a tangent vector. Hence *a developable surface is the locus of all tangent lines to a tortuous curve.*

Of course the tangent plane to the surface is the osculating plane at the corresponding point of the curve; and this is indicated by the fact that the normal to (1) is parallel to

$$V\phi' t\phi'' t. \quad \text{(See § 300.)}$$

*To find the form of the section of the surface made by a normal plane through a point in the curve.*

The equation of the surface in the neighbourhood of the extremity of $\rho$ is approximately

$$\varpi = \rho + s\rho' + \frac{s^2}{2}\rho'' + \&\text{c.} + x(\rho' + s\rho'' + \&\text{c.}).$$

The part of $\varpi - \rho$ which is parallel to $\rho'$ is

$$-\rho' S(\varpi - \rho)\rho' = -\rho'\left\{-(s+x) - \rho''^2\left(\frac{s^3}{6} + \frac{xs^2}{2}\right) + \ldots\right\};$$

therefore $\varpi - \rho = A\rho' + \left(\frac{s^2}{2} + xs\right)\rho'' - \left(\frac{s^3}{6} + \frac{xs^2}{2}\right)\rho' V\rho'\rho''' + \ldots.$

And, when $A = 0$, i.e. in the normal section, we have approximately

$$x = -s,$$

so that $$\varpi - \rho = -\frac{s^2}{2}\rho'' + \frac{s^3}{3}\rho' V\rho'\rho'''.$$

Hence the curve has an equation of the form

$$\sigma = s^2\alpha + s^3\beta,$$

a semicubical parabola.

312. A *Geodetic* line is a curve drawn on a surface so that its osculating plane at any point contains the normal to the surface. Hence, if $\nu$ be the normal at the extremity of $\rho$, $\rho'$ and $\rho''$ the first and second differentials of the vector of the geodetic,

$$S.\nu\rho'\rho'' = 0,$$

which may be easily transformed into

$$V.\nu dU\rho' = 0.$$

313. In the sphere $T\rho = a$ we have

$$\nu \parallel \rho,$$

hence
$$S . \rho\rho'\rho'' = 0,$$

which shews of course that $\rho$ is confined to a plane passing through the origin, the centre of the sphere.

For a formal proof, we may proceed as follows—. The above equation is equivalent to the three

$$S\theta\rho = 0, \qquad S\theta\rho' = 0, \qquad S\theta\rho'' = 0,$$

from which we see at once that $\theta$ is a constant vector, and therefore the first expression, which includes the others, is the complete integral.

Or we may proceed thus—

$$0 = -\rho S . \rho\rho'\rho'' + \rho'' S . \rho^2\rho' = V . V\rho\rho' V\rho\rho'' = V . V\rho\rho' dV\rho\rho',$$

whence by § 140 (2) we have at once

$$UV\rho\rho' = \text{const.} = \theta \text{ suppose,}$$

which gives the same results as before.

314. In any cone, when the vertex is taken as origin, we have, of course,

$$S\nu\rho = 0,$$

since $\rho$ lies in the tangent plane. But we have also

$$S\nu\rho' = 0.$$

Hence, by the general equation of § 312, eliminating $\nu$ we get

$$0 = S . \rho\rho' V\rho'\rho'' = S\rho dU\rho' \text{ by § 140 (2).}$$

Integrating $\quad C = S\rho U\rho' - \int Sd\rho U\rho' = S\rho U\rho' + \int Td\rho.$

The interpretation of this is, that the length of any arc of the geodetic is equal to the projection of the side of the cone (drawn to its extremity) upon the tangent to the geodetic. In other words, *when the cone is developed on a plane the geodetic becomes a straight line.* A similar result may easily be obtained for the geodetic lines on any developable surface whatever.

315. *To find the shortest line connecting two points on a given surface.*

Here $\int Td\rho$ is to be a minimum, subject to the condition that $d\rho$ lies in the given surface. [We employ $\delta$, though (in the notation we employ) it would naturally denote a vector, as the symbol of variation.]

Now
$$\delta\int Td\rho = \int \delta Td\rho = -\int \frac{Sd\rho d\delta\rho}{Td\rho} = -\int S.Ud\rho d\delta\rho$$
$$= -[S.Ud\rho\delta\rho] + \int S.\delta\rho dUd\rho,$$

where the term in brackets vanishes at the limits, as the extreme points are fixed, and therefore $\delta\rho = 0$ at each.

Hence our only conditions are

$$\int S.\delta\rho dUd\rho = 0, \quad \text{and} \quad S\nu\delta\rho = 0, \quad \text{giving}$$
$$V.\nu dUd\rho = 0, \text{ as in § 312.}$$

If the extremities of the curve are not given, but are to lie on given curves, we must refer to the integrated portion of the expression for the variation of the length of the arc. And its form

$$S.Ud\rho\delta\rho$$

shews that the shortest line cuts each of the given curves at right angles.

316. The osculating plane of the curve

$$\rho = \phi t$$

is
$$S.\phi' t\phi'' t(\varpi - \rho) = 0 \quad\text{.....................(1),}$$

and is, of course, the tangent plane to the surface

$$\rho = \phi t + u\phi' t \quad\text{.........................(2).}$$

Let us attempt the converse of the process we have, so far, pursued, and endeavour to find (2) as the envelop of the variable plane (1).

Differentiating (1) with respect to $t$ only, we have

$$S.\phi'\phi'''(\varpi - \rho) = 0.$$

By this equation, combined with (1), we have

$$\varpi - \rho \parallel V.V\phi'\phi'' V\phi'\phi''' \parallel \phi',$$

or
$$\varpi = \rho + u\phi' = \phi + u\phi',$$

which is equation (2).

317. This leads us to the consideration of envelops generally, and the process just employed may easily be extended to the problem of *finding the envelop of a series of surfaces whose equation contains one scalar parameter.*

When the given equation is a scalar one, the process of finding the envelop is precisely the same as that employed in ordinary Cartesian geometry, though the work is often shorter and simpler.

If the equation be given in the form

$$\rho = \psi(t, u, v),$$

where $\psi$ is a vector function, $t$ and $u$ the scalar variables for any one surface, $v$ the scalar parameter, we have for a proximate surface

$$\rho_1 = \psi(t_1, u_1, v_1) = \rho + \psi'_t \, \delta t + \psi'_u \, \delta u + \psi'_v \, \delta v.$$

Hence at all points on the intersection of two successive surfaces of the series we have

$$\psi'_t \, \delta t + \psi'_u \, \delta u + \psi'_v \, \delta v = 0,$$

which is equivalent to the following scalar equation connecting the quantities $t$, $u$ and $v$;

$$S \, . \, \psi'_t \psi'_u \psi'_v = 0.$$

This equation, along with

$$\rho = \psi(t, u, v),$$

enables us to eliminate $t$, $u$, $v$, and the resulting scalar equation is that of the required envelop.

318. As an example, let us find the envelop of the osculating plane of a tortuous curve. Here the equation of the plane is (§ 316)

$$S \, . \, (\varpi - \rho) \, \phi' t \phi'' t = 0,$$

or
$$\varpi = \phi t + x \phi' t + y \phi'' t = \psi(x, y, t),$$

if
$$\rho = \phi t$$

be the equation of the curve.

Our condition is, by last section,

$$S \, . \, \psi'_x \psi'_y \psi'_t = 0,$$

or
$$S \, . \, \phi' t \, \phi'' t \, [\phi' t + x \phi'' t + y \phi''' t] = 0,$$

or
$$y S \, . \, \phi' t \, \phi'' t \, \phi''' t = 0.$$

Now the second factor cannot vanish, unless the given curve be plane, so that we must have

$$y = 0,$$

and the envelop is
$$\varpi = \phi t + x \phi' t$$

the developable surface, of which the given curve is the edge of regression, as in § 316.

319. When the equation contains two scalar parameters, its differential coefficients with respect to them must vanish, and we have thus three equations from which to eliminate two numerical quantities.

A very common form in which these two scalar parameters appear in quaternions is that of an arbitrary unit-vector. In this case the problem may be thus stated:—

*Find the envelop of the surface whose scalar equation is*

$$F(\rho, \alpha) = 0,$$

*where $\alpha$ is subject to the one condition*

$$T\alpha = 1.$$

Differentiating with respect to $\alpha$ alone, we have

$$S\nu d\alpha = 0, \qquad S\alpha d\alpha = 0,$$

where $\nu$ is a known vector function of $\rho$ and $\alpha$. Since $d\alpha$ may have any of an infinite number of values, these equations shew that

$$V\alpha\nu = 0.$$

This is equivalent to two scalar conditions only, and these, in addition to the two given scalar equations, enable us to eliminate $\alpha$.

With the brief explanation we have given, and the examples which follow, the student will easily see how to treat any other set of data he may meet with in a question of envelops.

320. *Find the envelop of a plane whose distance from the origin is constant.*

Here $$S\alpha\rho = -c,$$

with the condition $$T\alpha = 1.$$

Hence, by last section, $$V\rho\alpha = 0,$$

and therefore $$\rho = c\alpha,$$

or $$T\rho = c,$$

the sphere of radius $c$, as was to be expected.

If we seek the *envelop of those only of the planes which are parallel to a given vector $\beta$*, we have the additional relation

$$S\alpha\beta = 0.$$

In this case the three differentiated equations are

$$S\rho d\alpha = 0, \qquad S\alpha d\alpha = 0, \qquad S\beta d\alpha = 0,$$

and they give $$S.\alpha\beta\rho = 0.$$

Hence $$\alpha = U.\beta V\beta\rho,$$

and the envelop is $$TV\beta\rho = cT\beta,$$

the circular cylinder of radius $c$ and axis coinciding with $\beta$.

By putting $S\alpha\beta = e$, where $e$ is a constant different from zero, we pick out all the planes of the series which have a definite inclination to $\beta$, and of course get as their envelop a right cone.

321. The equation $S^2\alpha\rho + 2S.\alpha\beta\rho = b$

represents a parabolic cylinder, whose generating lines are parallel to the vector $\alpha V\alpha\beta$. For the equation is of the second degree, and is not altered by increasing $\rho$ by the vector $x\alpha V\alpha\beta$; also the surface cuts planes perpendicular to $\alpha$ in one line, and planes perpendicular to $V\alpha\beta$ in two parallel lines. Its form and position of course depend upon the values of $\alpha$, $\beta$, and $b$. *It is required to find its envelop* if $\beta$ and $b$ be constant, and $\alpha$ be subject to the one scalar condition

$$T\alpha = 1.$$

The process of § 319 gives, by inspection,

$$\rho S\alpha\rho + V\beta\rho = x\alpha.$$

Operating by $S.\alpha$, we get

$$S^2\alpha\rho + S.\alpha\beta\rho = -x,$$

which gives $$S.\alpha\beta\rho = x + b.$$

But, by operating successively by $S.V\beta\rho$ and by $S.\rho$, we have

$$(V\beta\rho)^2 = xS.\alpha\beta\rho,$$

and $$(\rho^2 - x)\,S\alpha\rho = 0.$$

Omitting, for the present, the factor $S\alpha\rho$, these three equations give, by elimination of $x$ and $\alpha$,

$$(V\beta\rho)^2 = \rho^2\,(\rho^2 + b),$$

which is the equation of the envelop required.

This is evidently a surface of revolution of the fourth degree whose axis is $\beta$; but, to get a clearer idea of its nature, put

$$c^2\rho^{-1} = \varpi,$$

and the equation becomes

$$(V\beta\varpi)^2 = c^4 + b\varpi^2,$$

which is obviously a surface of revolution of the second degree, referred to its centre. Hence the required envelop is the *reciprocal* of such a surface, in the sense that *the rectangle under the lengths* *of condirectional radii of the two is constant:* i.e. it is the *Electric Image.*

We have a curious particular case if the constants are so related that

$$b + \beta^2 = 0,$$

for then the envelop breaks up into the two equal spheres, touching each other at the origin,

$$\rho^2 = \pm S\beta\rho,$$

while the corresponding surface of the second order becomes the two parallel planes

$$S\beta\varpi = \pm c^2.$$

322. The particular solution above met with, viz.

$$S\alpha\rho = 0,$$

limits the original problem, which now becomes one of finding the envelop of a line instead of a surface. In fact this equation, taken in conjunction with that of the parabolic cylinder, belongs to that generating line of the cylinder which is the locus of the vertices of the principal parabolic sections.

Our equations become

$$2S.\alpha\beta\rho = b,$$

$$S\alpha\rho = 0,$$

$$T\alpha = 1;$$

whence $$V\beta\rho = x\alpha;$$

giving $$x = -S.\alpha\beta\rho = -\frac{b}{2},$$

and thence $$TV\beta\rho = \frac{b}{2};$$

so that the envelop is a circular cylinder whose axis is $\beta$. [It is to be remarked that the equations above require that

$$S\alpha\beta = 0,$$

so that the problem now solved is merely that of *the envelop of a parabolic cylinder which rotates about its focal line.* This discussion has been entered into merely for the sake of explaining a peculiarity in a former result, because of course the present results can be obtained immediately by an exceedingly simple process.]

323. The equation $S\alpha\rho S.\alpha\beta\rho = a^2$,

with the condition $T\alpha = 1$,

represents a series of hyperbolic cylinders. *It is required to find their envelop.*

As before, we have

$$\rho S.\alpha\beta\rho + V\beta\rho S\alpha\rho = x\alpha,$$

which by operating by $S.\alpha$, $S.\rho$, and $S.V\beta\rho$, gives

$$2a^2 = -x,$$

$$\rho^2 S.\alpha\beta\rho = xS\alpha\rho,$$

$$(V\beta\rho)^2 S\alpha\rho = xS.\alpha\beta\rho.$$

Eliminating $\alpha$ and $x$ we have, as the equation of the envelop,

$$\rho^2 (V\beta\rho)^2 = 4a^4.$$

Comparing this with the equations

$$\rho^2 = -2a^2,$$

and $$(V\beta\rho)^2 = -2a^2,$$

which represent a sphere and one of its circumscribing cylinders, we see that, if condirectional radii of the three surfaces be drawn from the origin, that of the new surface is a geometric mean between those of the two others.

324. *Find the envelop of all spheres which touch one given line and have their centres in another.*

Let $$\rho = \beta + y\gamma$$

be the line touched by all the spheres, and let $x\alpha$ be the vector of the centre of any one of them, the equation is (by § 213, or § 214)

$$\gamma^2 (\rho - x\alpha)^2 = -\{V.\gamma(\beta - x\alpha)\}^2,$$

or, putting for simplicity, but without loss of generality,

$$T\gamma = 1, \qquad S\alpha\beta = 0, \qquad S\beta\gamma = 0,$$

so that $\beta$ is the least vector distance between the given lines,

$$(\rho - x\alpha)^2 = (\beta - x\alpha)^2 + x^2 S^2\alpha\gamma,$$

and, finally, $$\rho^2 - \beta^2 - 2xS\alpha\rho = x^2 S^2\alpha\gamma.$$

Hence, by § 317, $$-2S\alpha\rho = 2xS^2\alpha\gamma.$$

[This gives no definite envelop, except the point $\rho = \beta$, if

$$S\alpha\gamma = 0,$$

i.e. if the line of centres is perpendicular to the line touched by all the spheres.]

Eliminating $x$, we have for the equation of the envelop

$$S^2\alpha\rho + S^2\alpha\gamma(\rho^2 - \beta^2) = 0,$$

which denotes a surface of revolution of the second degree, whose axis is $\alpha$.

Since, from the form of the equation, $T\rho$ may have any magnitude not less than $T\beta$, and since the section by the plane

$$S\alpha\rho = 0$$

is a real circle, on the sphere

$$\rho^2 - \beta^2 = 0,$$

the surface is a hyperboloid of one sheet.

[It will be instructive to the student to find the signs of the values of $g_1$, $g_2$, $g_3$ as in § 177, and thence to prove the above conclusion.]

325. As a final example of this kind let us find the envelop of the hyperbolic cylinder

$$S\alpha\rho S\beta\rho - c = 0,$$

where the vectors $\alpha$ and $\beta$ are subject to the conditions

$$T\alpha = T\beta = 1,$$

$$S\alpha\gamma = 0, \qquad S\beta\delta = 0,$$

$\gamma$ and $\delta$ being given vectors.

[It will be easily seen that two of the six scalars involved in $\alpha$, $\beta$ still remain as variable parameters.]

We have $$S\alpha d\alpha = 0, \qquad S\gamma d\alpha = 0,$$

so that $$d\alpha = xV\alpha\gamma.$$

Similarly $$d\beta = yV\beta\delta.$$

But, by the equation of the cylinders,

$$S\alpha\rho S\rho d\beta + S\rho d\alpha S\beta\rho = 0,$$

or $$yS\alpha\rho S.\beta\delta\rho + xS.\alpha\gamma\rho S\beta\rho = 0.$$

Now by the nature of the given equation, neither $S\alpha\rho$ nor $S\beta\rho$ can vanish, so that the independence of $d\alpha$ and $d\beta$ requires

$$S.\alpha\gamma\rho = 0, \qquad S.\beta\delta\rho = 0.$$

Hence $$\alpha = U.\gamma V\gamma\rho, \qquad \beta = U.\delta V\delta\rho,$$

and the envelop is $$T.V\gamma\rho V\delta\rho - cT\gamma\delta = 0,$$

a surface of the fourth degree, which may be constructed by laying off mean proportionals between the lengths of condirectional radii of two equal right cylinders whose axes meet in the origin.

326. We may now easily see the truth of the following general statement.

Suppose the given equation of the series of surfaces, whose envelop is required, to contain $m$ vector, and $n$ scalar, parameters; and that these are subject to $p$ vector, and $q$ scalar, conditions.

In all there are $3m+n$ scalar parameters, subject to $3p+q$ scalar conditions.

That there may be an envelop we must therefore in general have

$$(3m+n)-(3p+q)=1, \quad \text{or} \quad =2.$$

In the former case the enveloping surface is given as the locus of a series of *curves*, in the latter of a series of *points*.

Differentiation of the equations gives us $3p+q+1$ equations, linear and homogeneous in the $3m+n$ differentials of the scalar parameters, so that by the elimination of these we have *one* final scalar equation in the first case, *two* in the second; and thus in each case we have just equations enough to eliminate all the arbitrary parameters.

Sometimes a very simple consideration renders laborious calculation unnecessary. Thus *a rectangular system turns about the centre of an ellipsoid. Find the envelop of the plane which passes through the three points of intersection.*

If $\alpha$, $\beta$, $\gamma$ be the rectangular unit-system, the points of intersection with

$$S\rho\phi\rho=-1$$

are at the extremities of

$$\frac{\alpha}{\sqrt{-S\alpha\phi\alpha}}, \quad \frac{\beta}{\sqrt{-S\beta\phi\beta}}, \quad \frac{\gamma}{\sqrt{-S\gamma\phi\gamma}}.$$

And if these lie in the plane

$$S\epsilon\rho=1 \quad \text{..............................(1)},$$

we must have

$$\epsilon=-\Sigma\alpha\sqrt{-S\alpha\phi\alpha} \quad \text{........................(2)}.$$

It would be troublesome to work out the envelop of (1), with (2) and the conditions of a rectangular unit-system as the data, but we may proceed as follows.

The length of the perpendicular from the centre on the plane (1) is

$$T\epsilon^{-1}=\frac{1}{\sqrt{-(S\alpha\phi\alpha+S\beta\phi\beta+S\gamma\phi\gamma)}},$$

a constant, by § 185. Hence the envelop is a sphere of which this is the radius; and it has the same property for all the ellipsoids which, having their axes in the same lines as the first, intersect it at the point on

$$i+j+k.$$

We may obtain the result in another way. By § 281 the sum of the reciprocals of the squares of three rectangular central vectors of an *ellipsoid* is constant; while it is easily shewn (see Ex. 20 to Chap. VII.) that the same sum with regard to a *plane* is the reciprocal of the square of its distance from the origin.

327. *To find the locus of the foot of the perpendicular drawn from the origin to a tangent plane to any surface.*

If $$S\nu d\rho=0$$ be the differentiated equation of the surface, the equation of the tangent plane is

$$S(\varpi-\rho)\nu=0.$$

We may introduce the condition

$$S\nu\rho=-1,$$

which in general alters the tensor of $\nu$, so that $-\nu^{-1}$ becomes the required vector perpendicular, as it satisfies the equation

$$S\varpi\nu=-1.$$

It remains that we eliminate $\rho$ between the equation of the given surface, and the vector equation

$$\varpi=-\nu^{-1}.$$

The result is the scalar equation (in $\varpi$) required.

For example, if the given surface be the ellipsoid

$$S\rho\phi\rho=-1,$$

we have $$\varpi^{-1}=-\nu=-\phi\rho,$$

so that the required equation is

$$S\varpi^{-1}\phi^{-1}\varpi^{-1}=-1,$$

or $$S\varpi\phi^{-1}\varpi=-\varpi^4,$$

which is Fresnel's *Surface of Elasticity*. (§ 278.)

It is well to remark that this equation is derived from that of the reciprocal ellipsoid

$$S\rho\phi^{-1}\rho=-1$$

by putting $\varpi^{-1}$ for $\rho$.

328. *To find the reciprocal of a given surface with respect to the unit sphere whose centre is the origin.*

With the condition $S\rho\nu=-1$,

of last section, we see that $\nu$ is the vector of the pole of the tangent plane

$$S(\varpi - \rho)\nu = 0.$$

Hence we must put $\varpi = \nu,$

and eliminate $\rho$ by the help of the equation of the given surface.

Take the ellipsoid of last section, and we have

$$\varpi = \phi\rho,$$

so that the reciprocal surface is represented by

$$S\varpi\phi^{-1}\varpi = -1.$$

It is obvious that the former ellipsoid can be produced from this by a second application of the process.

And the property is general, for

$$S\rho\nu = -1$$

gives, by differentiation, and attention to the condition

$$S\nu d\rho = 0$$

the new relation $S\rho d\nu = 0,$

so that $\rho$ and $\nu$ are corresponding vectors of the two surfaces: either being that of the pole of a tangent plane drawn at the extremity of the other.

329. If the given surface be a cone with its vertex at the origin, we have a peculiar case. For here every tangent plane passes through the origin, and therefore the required locus is wholly at an infinite distance. The difficulty consists in $S\rho\nu$ becoming in this case a numerical multiple of the quantity which is equated to zero in the equation of the cone, so that of course we cannot put as above

$$S\rho\nu = -1.$$

330. The properties of the normal vector $\nu$ enable us to write the partial differential equations of families of surfaces in a very simple form.

Thus the distinguishing property of *Cylinders* is that all their generating lines are parallel. Hence all positions of $\nu$ must be parallel to a given plane—or

$$S\alpha\nu = 0,$$

which is the quaternion form of the well-known equation

$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0.$$

To integrate it, remember that we have always

$$S\nu d\rho = 0,$$

and that as $\nu$ is perpendicular to $\alpha$ it may be expressed in terms of any two vectors, $\beta$ and $\gamma$, each perpendicular to $\alpha$.

Hence $$\nu = x\beta + y\gamma,$$

and $$xS\beta d\rho + yS\gamma d\rho = 0.$$

This shews that $S\beta\rho$ and $S\gamma\rho$ are together constant or together variable, so that

$$S\beta\rho = f(S\gamma\rho),$$

where $f$ is any scalar function whatever.

331. In *Surfaces of Revolution* the normal intersects the axis. Hence, taking the origin in the axis $\alpha$, we have

$$S.\alpha\rho\nu = 0,$$

or $$\nu = x\alpha + y\rho.$$

Hence $$xS\alpha d\rho + yS\rho d\rho = 0,$$

whence the integral $$T\rho = f(S\alpha\rho).$$

The more common form, which is easily derived from that just written, is

$$TV\alpha\rho = F(S\alpha\rho).$$

In *Cones* we have $$S\nu\rho = 0,$$

and therefore

$$S\nu d\rho = S.\nu(T\rho dU\rho + U\rho dT\rho) = T\rho S\nu dU\rho.$$

Hence $$S\nu dU\rho = 0,$$

so that $\nu$ must be a function of $U\rho$, and therefore the integral is

$$f(U\rho) = 0,$$

which simply expresses the fact that the equation does not involve the tensor of $\rho$, i.e. that in Cartesian cöordinates it is homogeneous.

332. *If equal lengths be laid off on the normals drawn to any surface, the new surface formed by their extremities is normal to the same lines.*

For we have $$\varpi = \rho + aU\nu,$$

and $$S\nu d\varpi = S\nu d\rho + aS\nu dU\nu = 0,$$

which proves the proposition.

Take, for example, the surface

$$S\rho\phi\rho = -1;$$

the above equation becomes

$$\varpi = \rho + \frac{a\phi\rho}{T\phi\rho};$$

so that

$$\rho = \left(\frac{a\phi}{T\phi\rho} + 1\right)^{-1} \varpi,$$

and the equation of the new surface is to be found by eliminating $\frac{a}{T\phi\rho}$ (written $x$) between the equations

$$-1 = S.(x\phi + 1)^{-1}\varpi\phi(x\phi + 1)^{-1}\varpi,$$

and

$$-\frac{a^2}{x^2} = S.\phi(x\phi + 1)^{-1}\varpi\phi(x\phi + 1)^{-1}\varpi.$$

333. It appears from last section that if one orthogonal surface can be drawn cutting a given system of straight lines, an indefinitely great number may be drawn: and that the portions of these lines intercepted between any two selected surfaces of the series are all equal.

Let

$$\rho = \sigma + x\tau,$$

where $\sigma$ and $\tau$ are vector functions of $\rho$, and $x$ is any scalar, be the general equation of a system of lines: we have

$$S\tau d\rho = 0 = S(\rho - \sigma)\,d\rho$$

as the differentiated equation of the series of orthogonal surfaces, if it exist. Hence the following problem.

334. It is required *to find the criterion of integrability of the equation*

$$S\nu d\rho = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

*as the complete differential of the equation of a series of surfaces.*

Hamilton has given (*Elements*, p. 702) an extremely elegant solution of this problem, by means of the properties of linear and vector functions. We adopt a different and somewhat less rapid process, on account of some results it offers which will be useful to us later; and also because it will shew the student the connection of our methods with those of ordinary differential equations.

If we assume

$$F\rho = C$$

to be the integral, we have by § 144,

$$Sd\rho\nabla F = 0.$$

Comparing with the given equation, (1), we see that the latter represents a series of surfaces if $\nu$, or a scalar multiple of it, can be expressed as $\nabla F$.

If $$\nu = \nabla F,$$

we have $$\nabla\nu = \nabla^2 F = -\left(\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}\right),$$

and the last-written quantities are necessarily scalars, so that the only requisite condition of the integrability of (1) is

$$V\nabla\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If $\nu$ do not satisfy this criterion, it may when multiplied by a scalar. Hence the farther condition

$$V\nabla(w\nu) = 0,$$

which may be written

$$V\nu\nabla w - wV\nabla\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This requires that

$$S\nu\nabla\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If then (2) be not satisfied, we must try (4). If (4) be satisfied $w$ will be found from (3); and in either case (1) is at once integrable.

[If we put $$d\nu = \phi d\rho,$$

where $\phi$ is a linear and vector function, not necessarily self-conjugate, we have

$$V\nabla\nu = V\left(i\frac{d\nu}{dx} + \ldots\right) = V(i\phi i + \ldots) = -\epsilon,$$

by § 185. Thus, if $\phi$ be self-conjugate, $\epsilon = 0$, and the criterion (2) is satisfied. If $\phi$ be not self-conjugate we have by (4) for the criterion

$$S\epsilon\nu = 0.$$

These results accord with Hamilton's, lately referred to, but the mode of obtaining them is quite different from his.]

335. As a simple example let us first take *lines diverging from a point.* Here $\nu \parallel \rho$, and we see that if $\nu = \rho$

$$\nabla\nu = -3,$$

so that (2) is satisfied. And the equation is

$$S\rho d\rho = 0,$$

whose integral $$T\rho^2 = C$$

gives a series of concentric spheres.

*Lines perpendicular to, and intersecting, a fixed line.*

If $\alpha$ be the fixed line, $\beta$ any of the others, we have

$$S.\alpha\beta\rho = 0, \qquad S\alpha\beta = 0, \qquad S\beta d\rho = 0.$$

Here $$\nu \parallel \alpha V\alpha\rho,$$
and therefore equal to it, because (2) is satisfied.

Hence $$S \,.\, d\rho\alpha V\alpha\rho = 0,$$
or $$S \,.\, V\alpha\rho V\alpha d\rho = 0,$$
whose integral is the equation of a series of right cylinders
$$T^2 V\alpha\rho = C.$$

*To find the orthogonal trajectories of a series of circles whose centres are in, and their planes perpendicular to, a given line.*

Let $\alpha$ be a unit-vector in the direction of the line, then one of the circles has the equations
$$\left.\begin{array}{l} T\rho = C \\ S\alpha\rho = C' \end{array}\right\},$$
where $C$ and $C'$ are any constant scalars whatever.

Hence, for the required surfaces
$$\nu \parallel d_1\rho \parallel V\alpha\rho,$$
where $d_1\rho$ is an element of one of the circles, $\nu$ the normal to the orthogonal surface. Now let $d\rho$ be an element of a tangent to the orthogonal surface, and we have
$$S\nu d\rho = S \,.\, \alpha\rho d\rho = 0.$$
This shews that $d\rho$ is in the same plane as $\alpha$ and $\rho$, i.e. that the orthogonal surfaces are planes passing through the common ax

[To integrate the equation
$$S \,.\, \alpha\rho d\rho = 0$$
evidently requires, by § 334, the introduction of a factor. For
$$\begin{aligned} V \,.\, \nabla V\alpha\rho &= -\alpha S\nabla\rho + S\alpha\nabla \,.\, \rho \quad (\S\ 90, (1)) \\ &= 2\alpha, \end{aligned}$$
so that the first criterion is not satisfied. But
$$S \,.\, V\alpha\rho V \,.\, \nabla V\alpha\rho = 2S \,.\, \alpha V\alpha\rho = 0,$$
so that the second criterion holds. It gives, by (3) of § 334,
$$- V \,.\, V\alpha\rho\nabla w + 2w\alpha = 0,$$
or $$\rho S\alpha\nabla w - \alpha S\rho\nabla w + 2w\alpha = 0.$$

That is $$\left.\begin{array}{l} S\alpha\nabla w = 0 \\ S\rho\nabla w = 2w \end{array}\right\}.$$

These equations are satisfied by
$$w = \frac{1}{V^2\alpha\rho}.$$

But a simpler mode of integration is easily seen. Our equation may be written

$$0 = S.\alpha V\frac{d\rho}{\rho} = S\alpha\frac{dU\rho}{U\rho} = d.S\alpha\log U\frac{\rho}{\beta},$$

which is immediately integrable, $\beta$ being an arbitrary but constant vector.

As we have not introduced into this work the *logarithms* of versors, nor the corresponding *angles* of quaternions, we must refer to Hamilton's *Elements* for a further development of this point.]

**336.** As another example, let us *find series of surfaces which, together, divide space into cubes.*

If $\rho$ be the vector of one series which has the required property, $\sigma$ that of a second, it is clear that ($u$ being a scalar)

$$d\sigma = uq^{-1}d\rho q \text{..........................}(1),$$

where $u$ and $q$ are functions of $\rho$. For, to values of $d\rho$ belonging to edges of one cube correspond values of $d\sigma$ belonging to edges of another. Operate by $S.\alpha$, where $\alpha$ is any constant vector, then

$$S\alpha d\sigma = uS.q\alpha q^{-1}d\rho.$$

As the left-hand member is a complete differential, we have by § 334

$$.V\nabla(uq\alpha q^{-1}) = 0.$$

This is easily put in the form [Chap. IV., Ex. (4), and § 140 (8)]

$$V.\frac{\nabla u}{u}q\alpha q^{-1} = -2q\alpha q^{-1}S.\nabla q q^{-1} + 2S(q\alpha q^{-1}\nabla)q.q^{-1} \text{...}(2).$$

Multiply by $q\alpha q^{-1}$, and add together three equations of the resulting form, in which the values of $\alpha$ form a rectangular unit system. Then

$$+2\frac{\nabla u}{u} = +6S.\nabla q\,q^{-1} - 2\nabla q\,q^{-1}.$$

This shews that

$$S.\nabla q\,q^{-1} = 0.$$

Take account of this result in (2), and put $d\rho$ for $q\alpha q^{-1}$, which may be any vector. Thus

$$V.d\rho\frac{\nabla u}{u} = 2dq.q^{-1} = 2V\frac{dq}{q} \text{ ...............} (3).$$

From this we see at once that

$$q = aU\rho,$$

where $a$ is any constant versor. Then (3) gives

$$\frac{\nabla u}{u} = -\frac{2U\rho}{T\rho}, \text{ or } \frac{du}{u} = \frac{2S\rho d\rho}{T\rho^2},$$

so that

$$u = \frac{C}{T\rho^2}.$$

Thus, from (1), we have

$$\sigma = a\frac{C}{\rho}a^{-1} + \beta.$$

This gives the Electric Image transformation, with any subsequent rotation, followed (or, as is easily seen, preceded) by a translation. Hence the only series of surfaces which satisfy the question, are mutually perpendicular planes; and their images, which are series of spheres, passing through a common point and having their centres on three rectangular lines passing through that point.

[For another mode of solution see *Proc. R. S. E.*, Dec. 1877.]

In some respects analogous to this is the celebrated physical problem of finding series of *Orthogonal Isothermal Surfaces.* We give a slight sketch of it here.

If three such series of surfaces be denoted by their temperatures thus:—

$$F_1 = T_1, \ F_2 = T_2, \ F_3 = T_3,$$

the conditions of orthogonality are fully expressed by putting for the respective values of the flux of heat in each series

$$\nabla F_1 = u_1 q\alpha q^{-1}, \quad \nabla . F_2 = u_2 q\beta q^{-1}, \quad \nabla F_3 = u_3 q\gamma q^{-1},$$

where $\alpha$, $\beta$, $\gamma$ form a constant rectangular unit vector system.

But the isothermal conditions are simply

$$\nabla^2 F_1 = \nabla^2 F_2 = \nabla^2 F_3 = 0.$$

Hence we have three simultaneous equations, of which the first is

$$\nabla . u_1 q\alpha q^{-1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

[In the previous problem $\alpha$ might be any vector whatever, the values of $u$ were equal, and the vector part only of the left-hand member was equated to zero. These conditions led to an unique form of solution. Nothing of the kind is to be expected here.]

From equation $(a)$ we have at once

$$\left.\begin{array}{r} S . \alpha'\nabla v_1 + 2S . \alpha'\nabla q\, q^{-1} = 0 \\ V . \alpha'\nabla v_1 - 2\alpha' S . \nabla q\, q^{-1} + 2S(\alpha'\nabla)\, q\, q^{-1} = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (b)$$

where $v_1$ has been put for $\log u_1$, and $\alpha'$ for $q\alpha q^{-1}$.

From the group of six equations, of which (*b*) gives two, we have

$$S.\nabla q\, q^{-1} = 0;$$

with three of the type

$$S.\alpha' S(\alpha'\nabla)\, q q^{-1} = 0.$$

We also obtain without difficulty

$$\Sigma \nabla v + 4\nabla q\, q^{-1} = 0,$$

which may be put in the form

$$\nabla.(u_1 u_2 u_3)^{\frac{1}{4}}\, q = 0.$$

But when we attempt to find the value of $dq$ we are led to expressions such as

$$\Sigma V.\alpha' S(\alpha' d\rho)\nabla v_1 + 2dq\, q^{-1} = 0,$$

which, in consequence of the three different values of $v$, are comparatively unmanageable. (See Ex. 24 at end of Chapter.) They become, however, comparatively simple when one of the three families is assumed. The student will find it useful to work out the problem when $F_1$ represents a series of parallel planes, so that the others are cylinders; or when $F_1$ represents planes passing through a line, the others being surfaces of revolution; &c.

337. *To find the orthogonal trajectories of a given series of surfaces.*

If the equation $F\rho = C$

give $S\nu d\rho = 0,$

the equation of the orthogonal curves is

$$V\nu d\rho = 0.$$

This is equivalent to two scalar differential equations (§ 210), which, when the problem is possible, belong to surfaces on each of which the required lines lie. The finding of the requisite criterion we leave to the student. [He has only to operate on the last-written equation by $S.\alpha$, where $\alpha$ is *any* constant vector; and, bearing this italicized word in mind, proceed as in § 334.]

*Let the surfaces be concentric spheres.*

Here $\rho^2 = C,$

and therefore $V\rho d\rho = 0.$

Hence $T\rho^2 dU\rho = -U\rho V\rho d\rho = 0,$

and the integral is $U\rho = \text{constant},$

denoting straight lines through the origin.

*Let the surfaces be spheres touching each other at a common point.* The equation is (§ 235)

$$S\alpha\rho^{-1} = C,$$

whence $$V.\rho\alpha\rho d\rho = 0.$$

The integrals may be written

$$S.\alpha\beta\rho = 0, \qquad \rho^2 + hTV\alpha\rho = 0,$$

the first ($\beta$ being any vector) is a plane through the common diameter; the second represents a series of rings or *tores* (§ 340) formed by the revolution, about $\alpha$, of circles *touching* that line at the point common to the spheres.

*Let the surfaces be similar, similarly situated, and concentric, surfaces of the second degree.*

Here $$S\rho\chi\rho = C,$$

therefore $$V\chi\rho d\rho = 0.$$

But, by § 307, the integral of this equation is

$$\rho = e^{t\chi}\epsilon$$
$$= \phi^t\epsilon,$$

where $\phi$ and $\chi$ are related to each other, as in § 307; and $\epsilon$ is any constant vector.

338. *To integrate the linear partial differential equation of a family of surfaces.*

The equation (see § 330)

$$P\frac{du}{dx} + Q\frac{du}{dy} + R\frac{du}{dz} = 0$$

may be put in the very simple form

$$S(\sigma\nabla)u = 0 \text{..........................} (1),$$

if we write $$\sigma = iP + jQ + kR,$$

and $$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

[From this we see that the meaning of the differential equation is that *at every point of the surface*

$$u = const.$$

*the corresponding vector*, $\sigma$, *is a tangent line.* Thus we have a suggestion of the ordinary method of solving such equations.]

(1) gives, at once, $$\nabla u = mV\theta\sigma,$$

where $m$ is a scalar and $\theta$ a vector (in whose tensor $m$ might have been included, but is kept separate for a special purpose). Hence

$$\begin{aligned} du &= -S(d\rho\nabla)u \\ &= -mS.\theta\sigma d\rho \\ &= -S\theta d\tau, \end{aligned}$$

if we put
$$d\tau = mV\sigma d\rho$$

so that $m$ is an integrating factor of $V.\sigma d\rho$. If a value of $m$ can be found, it is obvious, from the form of the above equation, that $\theta$ must be a function of $\tau$ alone; and the integral is therefore

$$u = F(\tau) = \text{const.}$$

where $F$ is an arbitrary scalar function.

Thus the differential equation of *Cylinders* is

$$S(\alpha\nabla)u = 0,$$

where $\alpha$ is a constant vector. Here $m = 1$, and

$$u = F(V\alpha\rho) = \text{const.}$$

That of *Cones* referred to the vertex is

$$S(\rho\nabla)u = 0.$$

Here the expression to be made integrable is

$$V\rho d\rho.$$

But Hamilton long ago shewed that (§ 140 (2))

$$\frac{dU\rho}{U\rho} = V\frac{d\rho}{\rho} = \frac{V\rho d\rho}{(T\rho)^2},$$

which indicates the value of $m$, and gives

$$u = F(U\rho) = \text{const.}$$

It is obvious that the above is only one of a great number of different processes which may be applied to integrate the differential equation. It is quite easy, for instance, to pass from it to the assumption of a vector integrating factor instead of the scalar $m$, and to derive the usual criterion of integrability. There is no difficulty in modifying the process to suit the case when the right-hand member is a multiple of $u$. In fact it seems to throw a very clear light upon the whole subject of the integration of partial differential equations. If, instead of $S(\sigma\nabla)$, we employ other operators as $S(\sigma\nabla)S(\tau\nabla)$, $S.\sigma\nabla\tau\nabla$, &c. (where $\nabla$ may or may not operate on $u$ alone), we can pass to linear partial differential equations of the second and higher orders. Similar theorems can be obtained from vector operators, as $V(\sigma\nabla)$[*].

339. *Find the general equation of surfaces described by a line which always meets, at right angles, a fixed line.*

If $\alpha$ be the fixed line, $\beta$ and $\gamma$ forming with it a rectangular unit system, then

$$\rho = x\alpha + y(\beta + z\gamma),$$

where $y$ may have all values, but $x$ and $z$ are mutually dependent, is one form of the equation.

Another, expressing the arbitrary relation between $x$ and $z$, is

$$\frac{S\gamma\rho}{S\beta\rho} = f(S\alpha\rho).$$

But we may also write

$$\rho = \alpha F(x) + y\alpha^x\beta,$$

as it obviously expresses the same conditions.

The simplest case is when $F(x) = hx$. The surface is one which cuts, in a right helix, every cylinder which has $\alpha$ for its axis.

340. *The centre of a sphere moves in a given circle, find the equation of the ring described.*

Let $\alpha$ be the unit-vector axis of the circle, its centre the origin, $r$ its radius, $a$ that of the sphere.

Then $$(\rho - \beta)^2 = -a^2$$

is the equation of the sphere in any position, where

$$S\alpha\beta = 0, \qquad T\beta = r.$$

These give (§ 326) $S.\alpha\beta\rho = 0$, and $\beta$ must now be eliminated. The immediate result is that

$$\beta = r\alpha UV\alpha\rho,$$

giving $$(\rho^2 - r^2 + a^2)^2 = 4r^2T^2V\alpha\rho,$$
$$= 4r^2(-\rho^2 - S^2\alpha\rho),$$

which is the required equation. It may easily be changed to

$$(\rho^2 - a^2 + r^2)^2 = -4a^2\rho^2 - 4r^2S^2\alpha\rho \ldots\ldots\ldots\ldots\ldots(1),$$

and in this form it enables us to give a very simple proof of the singular property of the ring (or *tore*) discovered by Villarceau.

For the planes $$S\rho\left(\alpha \pm \frac{a\beta}{r\sqrt{r^2 - a^2}}\right) = 0,$$

---

[*] Tait, *Proc. R. S. E.*, 1869–70.

which together are represented by

$$r^2(r^2-a^2)S^2\alpha\rho - a^2S^2\beta\rho = 0,$$

evidently pass through the origin and touch (and cut) the ring.

The latter equation may be written

$$r^2S^2\alpha\rho - a^2(S^2\alpha\rho + S^2\rho U\beta) = 0,$$

or

$$r^2S^2\alpha\rho + a^2(\rho^2 + S^2.\alpha\rho U\beta) = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

The plane intersections of (1) and (2) lie obviously on the new surface

$$(\rho^2 - a^2 + r^2)^2 = 4a^2S^2.\alpha\rho U\beta,$$

which consists of two spheres of radius $r$, as we see by writing its separate factors in the form

$$(\rho \pm a\alpha U\beta)^2 + r^2 = 0.$$

341. It may be instructive to work out this problem from a different point of view, especially as it affords excellent practice in transformations.

*A circle revolves about an axis passing within it, the perpendicular from the centre on the axis lying in the plane of the circle: shew that, for a certain position of the axis, the same solid may be traced out by a circle revolving about an external axis in its own plane.*

Let $a = \sqrt{b^2+c^2}$ be the radius of the circle, $i$ the vector axis of rotation, $-c\alpha$ (where $T\alpha = 1$) the vector perpendicular from the centre on the axis $i$, and let the vector

$$bi + ci\alpha$$

be perpendicular to the plane of the circle.

The equations of the circle are

$$\left.\begin{aligned}(\rho - c\alpha)^2 + b^2 + c^2 = 0\\ S\left(i + \frac{c}{b}\,i\alpha\right)\rho = 0\end{aligned}\right\}.$$

Also

$$\begin{aligned}-\rho^2 &= S^2i\rho + S^2\alpha\rho + S^2.i\alpha\rho,\\ &= S^2i\rho + S^2\alpha\rho + \frac{b^2}{c^2}S^2i\rho\end{aligned}$$

by the second of the equations of the circle. But, by the first,

$$(\rho^2+b^2)^2 = 4c^2S^2\alpha\rho = -4(c^2\rho^2 + a^2S^2i\rho),$$

which is easily transformed into

$$(\rho^2 - b^2)^2 = -4a^2(\rho^2 + S^2i\rho),$$

or

$$\rho^2 - b^2 = -2aTVi\rho.$$

If we put this in the forms

$$\rho^2 - b^2 = 2aS\beta\rho,$$

and

$$(\rho - a\beta)^2 + c^2 = 0,$$

where $\beta$ is a unit-vector perpendicular to $i$ and in the plane of $i$ and $\rho$, we see at once that the surface will be traced out by, a circle of radius $c$, revolving about $i$, an axis in its own plane, distant $a$ from its centre.

[This problem is not well adapted to shew the gain in brevity and distinctness which generally attends the use of quaternions; as, from its very nature, it hints at the adoption of rectangular axes and scalar equations for its treatment, so that the solution we have given is but little different from an ordinary Cartesian one.]

342. *A surface is generated by a straight line which intersects two fixed straight lines: find the general equation.*

If the given lines intersect, there is no surface but the plane containing them.

Let then their equations be,

$$\rho = \alpha + x\beta, \quad \rho = \alpha_1 + x_1\beta_1.$$

Hence every point of the surface satisfies the condition, § 30,

$$\rho = y(\alpha + x\beta) + (1 - y)(\alpha_1 + x_1\beta_1) \ldots\ldots\ldots\ldots\ldots (1).$$

Obviously $y$ may have any value whatever: so that to specify a particular surface we must have a relation between $x$ and $x_1$. By the help of this, $x_1$ may be eliminated from (1), which then takes the usual form of the equation of a surface

$$\rho = \phi(x, y).$$

Or we may operate on (1) by $V.(\alpha + x\beta - \alpha_1 - x_1\beta_1)$, so that we get a vector equation equivalent to *two* scalar equations (§§ 98, 123), and not containing $y$. From this $x$ and $x_1$ may easily be found in terms of $\rho$, and the general equation of the possible surfaces may be written

$$f(x, x_1) = 0,$$

where $f$ is an arbitrary scalar function, and the values of $x$ and $x_1$ are expressed in terms of $\rho$.

This process is obviously applicable if we have, instead of two straight lines, any two given curves through which the line must pass; and even when the tracing line is itself a given curve, situated in a given manner. But an example or two will make the whole process clear.

343. *Suppose the moveable line to be restricted by the condition that it is always parallel to a fixed plane.*

Then, in addition to (1), we have the condition

$$S\gamma(\alpha_1 + x_1\beta_1 - \alpha - x\beta) = 0,$$

$\gamma$ being a vector perpendicular to the fixed plane.

We lose no generality by assuming $\alpha$ and $\alpha_1$, which are any vectors drawn from the origin to the fixed lines, to be each perpendicular to $\gamma$; for, if for instance we could not assume $S\gamma\alpha = 0$, it would follow that $S\gamma\beta = 0$, and the required surface would either be impossible, or would be a plane, cases which we need not consider. Hence

$$x_1 S\gamma\beta_1 - xS\gamma\beta = 0.$$

Eliminating $x_1$, by the help of this equation, from (1) of last section, we have

$$\rho = y(\alpha + x\beta) + (1-y)\left(\alpha_1 + x\beta_1 \frac{S\gamma\beta}{S\gamma\beta_1}\right).$$

Operating by any three non-coplanar vectors and with the characteristic $S$, we obtain three equations from which to eliminate $x$ and $y$. Operating by $S.\gamma$ we find

$$S\gamma\rho = xS\beta\gamma.$$

Eliminating $x$ by means of this, we have finally

$$S.\rho\left(\alpha + \frac{\beta S\gamma\rho}{S\beta\gamma}\right)\left(\alpha_1 + \frac{\beta_1 S\gamma\rho}{S\beta_1\gamma}\right) = 0,$$

which appears to be of the third degree. It is really, however, only of the second degree: since, in consequence of our assumptions, we have

$$V\alpha\alpha_1 \parallel \gamma,$$

and therefore $S\gamma\rho$ is a spurious factor of the left-hand side.

344. *Let the fixed lines be perpendicular to each other, and let the moveable line pass through the circumference of a circle, whose centre is in the common perpendicular, and whose plane bisects that line at right angles.*

Here the equations of the fixed lines may be written

$$\rho = \alpha + x\beta, \quad \rho = -\alpha + x_1\gamma,$$

where $\alpha$, $\beta$, $\gamma$ form a rectangular system, and we may assume the two latter to be unit-vectors.

The circle has the equations

$$\rho^2 = -a^2, \quad S\alpha\rho = 0.$$

Equation (1) of § 342 becomes

$$\rho = y(\alpha + x\beta) + (1-y)(-\alpha + x_1\gamma).$$

Hence $$S\alpha^{-1}\rho = y - (1-y) = 0, \text{ or } y = \tfrac{1}{2}.$$

Also $$\rho^2 = -\alpha^2 = (2y-1)^2\alpha^2 - x^2y^2 - x_1^2(1-y)^2,$$

or $$4a^2 = (x^2 + x_1^2),$$

so that if we now suppose the tensors of $\beta$ and $\gamma$ to be each $2a$, we may put $x = \cos\theta$, $x_1 = \sin\theta$, from which

$$\rho = (2y-1)\alpha + y\beta\cos\theta + (1-y)\gamma\sin\theta;$$

and finally $$\frac{S^2\beta\rho}{(1+S\alpha^{-1}\rho)^2} + \frac{S^2\gamma\rho}{(1-S\alpha^{-1}\rho)^2} = 4a^4.$$

For this specially simple case the solution is not better than the ordinary Cartesian one; but the student will easily see that we may by very slight changes adapt the above to data far less symmetrical than those from which we started. Suppose, for instance, $\beta$ and $\gamma$ not to be at right angles to one another; and suppose the plane of the circle not to be parallel to their plane, &c., &c. But farther, operate on every line in space by the linear and vector function $\phi$, and we distort the circle into an ellipse, the straight lines remaining straight. If we choose a form of $\phi$ whose principal axes are parallel to $\alpha$, $\beta$, $\gamma$, the data will remain symmetrical, but not unless. This subject will be considered again in the next Chapter.

345. *To find the curvature of a normal section of a central surface of the second degree.*

In this, and the few similar investigations which follow, it will be simpler to employ infinitesimals than differentials; though for a thorough treatment of the subject the latter method, as it may be seen in Hamilton's *Elements*, is preferable.

We have, of course, $$S\rho\phi\rho = -1,$$

and, if $\rho + \delta\rho$ be also a vector of the surface, we have rigorously, *whatever be the tensor of* $\delta\rho$,

$$S(\rho + \delta\rho)\phi(\rho + \delta\rho) = -1.$$

Hence $$2S\delta\rho\phi\rho + S\delta\rho\phi\delta\rho = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $\phi\rho$ is normal to the tangent plane at the extremity of $\rho$, so that if $t$ denote the distance of the point $\rho + \delta\rho$ from that plane

$$t = -S\delta\rho U\phi\rho,$$

and (1) may therefore be written

$$2tT\phi\rho - T^2\delta\rho S \,.\, U\delta\rho\phi U\delta\rho = 0.$$

But the curvature of the section is evidently

$$\mathcal{L}\,\frac{2t}{T^2\delta\rho},$$

or, by the last equation,

$$\frac{1}{T\phi\rho}\,\mathcal{L}S \,.\, U\delta\rho\phi U\delta\rho.$$

In the limit, $\delta\rho$ is a vector in the tangent plane; let $\varpi$ be the vector semidiameter of the surface which is parallel to it, and the equation of the surface gives

$$T^2\varpi \,.\, S \,.\, U\varpi\phi U\varpi = -1,$$

so that the curvature of the normal section, at the point $\rho$, in the direction of $\varpi$, is

$$\frac{1}{T\phi\rho T^2\varpi},$$

*directly as the perpendicular from the centre on the tangent plane, and inversely as the square of the semidiameter parallel to the tangent line*, a well-known theorem.

346. By the help of the known properties of the central section parallel to the tangent plane, this theorem gives us all the ordinary properties of the directions of maximum and minimum curvature, their being at right angles to each other, the curvature in any normal section in terms of the chief curvatures and the inclination to their planes, &c., &c., without farther analysis. And when, in a future section, we shew how to find an *osculating* surface of the second degree at any point of a given surface, the same properties will be at once established for surfaces in general. Meanwhile we may prove another curious property of the surfaces of the second degree, which similar reasoning extends to all surfaces.

The equation of the normal at the point $\rho + \delta\rho$ on the surface treated in last section is

$$\varpi = \rho + \delta\rho + x\phi(\rho + \delta\rho) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

This intersects the normal at $\rho$ if (§§ 216, 223)

$$S \,.\, \delta\rho\phi\rho\phi\delta\rho = 0,$$

that is, by the result of § 290, if $\delta\rho$ be parallel to the maximum or minimum diameter of the central section parallel to the tangent plane.

Let $\sigma_1$ and $\sigma_2$ be those diameters, then we may write in general

$$\delta\rho = p\sigma_1 + q\sigma_2,$$

where $p$ and $q$ are scalars, infinitely small.

If we draw through a point $P$ in the normal at $\rho$ a line parallel to $\sigma_1$, we may write its equation

$$\varpi = \rho + a\phi\rho + y\sigma_1.$$

The proximate normal (1) passes this line at a distance (see § 216)

$$S.(a\phi\rho - \delta\rho)\, UV\sigma_1\phi(\rho + \delta\rho),$$

or, neglecting terms of the second order,

$$\frac{1}{TV\sigma_1\phi\rho}(apS.\phi\rho\sigma_1\phi\sigma_1 + aqS.\phi\rho\sigma_1\phi\sigma_2 + qS.\sigma_1\sigma_2\phi\rho).$$

The first term in the bracket vanishes because $\sigma_1$ is a principal vector of the section parallel to the tangent plane, and thus the expression becomes

$$q\left(\frac{a}{T\sigma_2} - T\sigma_2\right).$$

Hence, if we take $a = T\sigma_2^2$, the distance of the normal from the new line is of the second order only. This makes the distance of $P$ from the point of contact $T\phi\rho T\sigma_2^2$, i.e. the principal radius of curvature along the tangent line parallel to $\sigma_2$. That is, *the group of normals drawn near a point of a central surface of the second degree pass ultimately through two lines each parallel to the tangent to one principal section, and drawn through the centre of curvature of the other.* The student may form a notion of the nature of this proposition by considering a small square plate, with normals drawn at every point, to be slightly bent, but by different amounts, in planes perpendicular to its edges. The first bending will make all the normals pass through the axis of the cylinder of which the plate now forms part; the second bending will not sensibly disturb this arrangement, except by lengthening or shortening the line in which the normals meet, but it will make them meet also in the axis of the new cylinder, at right angles to the first. A small pencil of light, with its focal lines, presents this appearance, due to the fact that a series of rays originally normal to a surface remain normals to a surface after any number of reflections and (ordinary) refractions. (See § 332.)

347. To extend these theorems to surfaces in general, it is only necessary, as Hamilton has shewn, to prove that if we write

$$d\nu = \phi d\rho,$$

$\phi$ is a *self-conjugate* function; and then the properties of $\phi$, as explained in preceding Chapters, are applicable to the question.

As the reader will easily see, this is merely another form of the investigation contained in § 334. But it is again cited here to shew what a number of very simple demonstrations may be given of almost all quaternion theorems.

The vector $\nu$ is defined by an equation of the form

$$df\rho = S\nu d\rho,$$

where $f$ is a scalar function. Operating on this by another independent symbol of differentiation, $\delta$, we have

$$\delta df\rho = S\delta\nu d\rho + S\nu\delta d\rho.$$

In the same way we have

$$d\delta f\rho = Sd\nu\delta\rho + S\nu d\delta\rho.$$

But, as $d$ and $\delta$ are independent, the left-hand members of these equations, as well as the second terms on the right (if these exist at all), are equal, so that we have

$$Sd\nu\delta\rho = S\delta\nu d\rho.$$

This becomes, putting $$d\nu = \phi d\rho,$$
and therefore $$\delta\nu = \phi\delta\rho,$$
$$S\delta\rho\phi d\rho = Sd\rho\phi\delta\rho,$$

which proves the proposition.

348. If we write the differential of the equation of a surface in the form

$$df\rho = 2S\nu d\rho,$$

then it is easy to see that

$$f(\rho + d\rho) = f\rho + 2S\nu d\rho + Sd\nu d\rho + \&c.,$$

the remaining terms containing as factors the third and higher powers of $Td\rho$. To the second order, then, we may write, except for certain singular points,

$$0 = 2S\nu d\rho + Sd\nu d\rho,$$

and, as before, (§ 345), the curvature of the normal section whose tangent line is $d\rho$ is

$$\frac{1}{T\nu} S \frac{d\nu}{d\rho}.$$

349. The step taken in last section, although a very simple one, virtually implies that the first three terms of the expansion of $f(\rho+d\rho)$ are to be formed in accordance with Taylor's Theorem, whose applicability to the expansion of scalar functions of quaternions has not been proved in this work (see § 142); we therefore give another investigation of the curvature of a normal section, employing for that purpose the formulae of § 299.

We have, treating $d\rho$ as an element of a curve,

$$S\nu d\rho = 0,$$

or, making $s$ the independent variable,

$$S\nu\rho' = 0.$$

From this, by a second differentiation,

$$S\frac{d\nu}{ds}\rho' + S\nu\rho'' = 0.$$

The curvature is, therefore, since $\nu \parallel \rho''$ and $T\rho' = 1$,

$$T\rho'' = -\frac{1}{T\nu}S\frac{d\nu}{d\rho}\rho'^2 = \frac{1}{T\nu}S\frac{d\nu}{d\rho}, \text{ as before.}$$

350. Since we have shewn that

$$d\nu = \phi d\rho$$

where $\phi$ is a self-conjugate linear and vector function, whose constants depend only upon the nature of the surface, and the position of the point of contact of the tangent plane; so long as we do not alter these we must consider $\phi$ as possessing the properties explained in Chapter V.

Hence, as the expression for $T\rho''$ does not involve the tensor of $d\rho$, we may put for $d\rho$ any unit-vector $\tau$, subject of course to the condition

$$S\nu\tau = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

And the curvature of the normal section whose tangent is $\tau$ is

$$\frac{1}{T\nu}S\frac{\phi\tau}{\tau} = -\frac{1}{T\nu}S\tau\phi\tau.$$

If we consider the central section of the surface of the second degree

$$S\varpi\phi\varpi + T\nu = 0,$$

made by the plane

$$S\nu\varpi = 0,$$

we see at once that the *curvature of the given surface along the normal section touched by* $\tau$ *is inversely as the square of the parallel radius in the auxiliary surface.* This, of course, includes Euler's and other well-known Theorems.

351. *To find the directions of maximum and minimum curvature,* we have

$$S\tau\phi\tau = \text{max. or min.}$$

with the conditions,
$$S\nu\tau = 0,$$
$$T\tau = 1.$$

By differentiation, as in § 290, we obtain the farther equation

$$S.\nu\tau\phi\tau = 0 \text{.........................} (1).$$

If $\tau$ be one of the two required directions, $\tau' = \tau U\nu$ is the other, for the last-written equation may be put in the form

$$S.\tau U\nu\phi\,(\nu\tau U\nu) = 0,$$

i.e.
$$S.\tau'\phi\,(\nu\tau') = 0,$$

or
$$S.\nu\tau'\phi\tau' = 0.$$

*Hence the sections of greatest and least curvature are perpendicular to one another.*

We easily obtain, as in § 290, the following equation

$$S.\nu\,(\phi + S\tau\phi\tau)^{-1}\,\nu = 0,$$

which gives two values of $S\tau\phi\tau$, and these divided by $-T\nu$ are the required curvatures.

352. Before leaving this very brief introduction to a subject, an exhaustive treatment of which will be found in Hamilton's *Elements*, we may make a remark on equation (1) of last section

$$S.\nu\tau\phi\tau = 0,$$

or, as it may be written, by returning to the notation of § 350,

$$S.\nu d\rho d\nu = 0.$$

*This is the general equation of lines of curvature.* For, if we define a line of curvature on any surface as a line such that normals drawn at contiguous points in it intersect, then, $\delta\rho$ being an element of such a line, the normals

$$\varpi = \rho + x\nu \quad \text{and} \quad \varpi = \rho + \delta\rho + y\,(\nu + \delta\nu)$$

must intersect. This gives, by § 216, the condition

$$S.\delta\rho\nu\delta\nu = 0,$$

as above.

## EXAMPLES TO CHAPTER X.

1. Find the length of any arc of a curve drawn on a sphere so as to make a constant angle with a fixed diameter.

2. Shew that, if the normal plane of a curve always contains a fixed line, the curve is a circle.

3. Find the radius of spherical curvature of the curve

$$\rho = \phi t.$$

Also find the equation of the locus of the centre of spherical curvature.

4. (Hamilton, *Bishop Law's Premium Examination,* 1854.)

(*a*) If $\rho$ be the variable vector of a curve in space, and if the differential $d\kappa$ be treated as $=0$, then the equation

$$dT(\rho - \kappa) = 0$$

obliges $\kappa$ to be the vector of some point in the normal plane to the curve.

(*b*) In like manner the system of two equations, where $d\kappa$ and $d^2\kappa$ are each $=0$,

$$dT(\rho - \kappa) = 0, \qquad d^2T(\rho - \kappa) = 0,$$

represents the axis of the element, or the right line drawn through the centre of the osculating circle, perpendicular to the osculating plane.

(*c*) The system of the three equations, in which $\kappa$ is treated as constant,

$$dT(\rho - \kappa) = 0, \qquad d^2T(\rho - \kappa) = 0, \qquad d^3T(\rho - \kappa) = 0,$$

determines the vector $\kappa$ of the centre of the osculating sphere.

(*d*) For the three last equations we may substitute the following:

$$S.(\rho - \kappa)\,d\rho = 0,$$

$$S.(\rho - \kappa)\,d^2\rho + d\rho^2 = 0,$$

$$S.(\rho - \kappa)\,d^3\rho + 3S.d\rho d^2\rho = 0.$$

(*e*) Hence, generally, whatever the independent and scalar variable may be, on which the variable vector $\rho$ of the curve depends, the vector $\kappa$ of the centre of the osculating sphere admits of being thus expressed:

$$\kappa = \rho + \frac{3V.d\rho d^2\rho S.d\rho d^2\rho - d\rho^2 V.d\rho d^3\rho}{S.d\rho d^2\rho d^3\rho}.$$

(*f*) In general

$$d(\rho^{-2}V.d\rho U\rho) = d(T\rho^{-3}V.\rho d\rho)$$
$$= T\rho^{-5}(3V.\rho d\rho S.\rho d\rho - \rho^2 V.\rho d^2\rho);$$

whence,

$$3V.\rho d\rho S.\rho d\rho - \rho^2 V.\rho d^2\rho = \rho^4 T\rho d(\rho^{-2}V.d\rho U\rho);$$

and, therefore, the recent expression for $\kappa$ admits of being thus transformed,

$$\kappa = \rho + \frac{d\rho^4 d(d\rho^{-2}V.d^2\rho U d\rho)}{S.d^2\rho d^3\rho U d\rho}.$$

(*g*) If the length of the element of the curve be constant, $dTd\rho = 0$, this last expression for the vector of the centre of the osculating sphere to a curve of double curvature becomes, more simply,

$$\kappa = \rho + \frac{d.d^2\rho d\rho^3}{S.d\rho d^2\rho d^3\rho};$$

or

$$\kappa = \rho + \frac{V.d^3\rho d\rho^3}{S.d\rho d^2\rho d^3\rho}.$$

(*h*) Verify that this expression gives $\kappa = 0$, for a curve described on a sphere which has its centre at the origin of vectors; or shew that whenever $dT\rho = 0$, $d^2T\rho = 0$, $d^3T\rho = 0$, as well as $dTd\rho = 0$, then

$$\rho S.d\rho^{-1}d^2\rho d^3\rho = V.d\rho d^3\rho.$$

5. Find the curve from every point of which three given spheres appear of equal magnitude.

6. Shew that the locus of a point, the difference of whose distances from each two of three given points is constant, is a plane curve.

7. Find the equation of the curve which cuts at a given angle all the sides of a cone of the second degree.

Find the length of any arc of this curve in terms of the distances of its extremities from the vertex.

8. Why is the centre of spherical curvature, of a curve described on a sphere, not necessarily the centre of the sphere?

9. Find the equation of the developable surface whose generating lines are the intersections of successive normal planes to a given tortuous curve.

10. Find the length of an arc of a tortuous curve whose normal planes are equidistant from the origin.

11. The reciprocals of the perpendiculars from the origin on the tangent planes to a developable surface are vectors of a tortuous curve; from whose osculating planes the cusp-edge of the original surface may be reproduced by the same process.

12. The equation $\rho = V\alpha^t\beta$,

where $\alpha$ is a unit-vector not perpendicular to $\beta$, represents an ellipse. If we put $\gamma = V\alpha\beta$, shew that the equations of the locus of the centre of curvature are

$$S.\beta\gamma\rho = 0,$$

$$S^{\frac{2}{3}}\beta\rho + S^{\frac{2}{3}}\gamma\rho = (\beta SU\alpha\beta)^{\frac{4}{3}}.$$

13. Find the radius of absolute curvature of a spherical conic.

14. If a cone be cut in a circle by a plane perpendicular to a side, the axis of the right cone which osculates it, along that side, passes through the centre of the section.

15. Shew how to find the vector of an umbilicus. Apply your method to the surfaces whose equations are

$$S\rho\phi\rho = -1,$$

and
$$S\alpha\rho S\beta\rho S\gamma\rho = -1.$$

16. Find the locus of the umbilici of the surfaces represented by the equation

$$S\rho\,(\phi + h)^{-1}\rho = -1,$$

where $h$ is an arbitrary parameter.

17. Shew how to find the equation of a tangent plane which touches a surface along a line, straight or curved. Find such planes for the following surfaces

$$S\rho\phi\rho = -1,$$

$$S\rho\,(\phi - \rho^2)^{-1}\rho = -1,$$

and
$$(\rho^2 - a^2 + b^2)^2 + 4\,(a^2\rho^2 + b^2S^2\alpha\rho) = 0.$$

18. Find the condition that the equation

$$S(\rho+\alpha)\phi\rho=-1,$$

where $\phi$ is a self-conjugate linear and vector function, may represent a cone.

19. Shew from the general equation that cones and cylinders are the only developable surfaces of the second degree.

20. Find the equation of the envelop of planes drawn at each point of an ellipsoid perpendicular to the radius vector from the centre.

21. Find the equation of the envelop of spheres whose centres lie on a given sphere, and which pass through a given point.

22. Find the locus of the foot of the perpendicular from the centre to the tangent plane of a hyperboloid of one, or of two, sheets.

23. Hamilton, *Bishop Law's Premium Examination*, 1852.

(*a*) If $\rho$ be the vector of a curve in space, the length of the element of that curve is $Td\rho$; and the variation of the length of a finite arc of the curve is

$$\delta\int Td\rho=-\int SUd\rho\delta d\rho=-\Delta SUd\rho\delta\rho+\int SdUd\rho\delta\rho.$$

(*b*) Hence, if the curve be a shortest line on a given surface, for which the normal vector is $\nu$, so that $S\nu\delta\rho=0$, this shortest or geodetic curve must satisfy the differential equation,

$$V\nu dUd\rho=0.$$

Also, for the extremities of the arc, we have the limiting equations,

$$SUd\rho_0\delta\rho_0=0; \qquad SUd\rho_1\delta\rho_1=0.$$

Interpret these results.

(*c*) For a spheric surface, $V\nu\rho=0$, $V.\rho dUd\rho=0$; the integrated equation of the geodetics is $V\rho Ud\rho=\varpi$, giving $S\varpi\rho=0$ (great circle).

For an arbitrary cylindric surface,

$$S\alpha\nu=0, \qquad S.\alpha dUd\rho=0;$$

the integral shews that the geodetic is generally a helix, making a constant angle with the generating lines of the cylinder.

(*d*) For an arbitrary conic surface,

$$S\nu\rho=0, \qquad S\rho dUd\rho=0;$$

integrate this differential equation, so as to deduce from it, $TV\rho U d\rho = \text{const.}$

Interpret this result; shew that the perpendicular from the vertex of the cone on the tangent to a given geodetic line is constant; this gives the rectilinear development.

When the cone is of the second degree, the same property is a particular case of a theorem respecting confocal surfaces.

(*e*) For a surface of revolution,

$$S.\alpha\rho\nu = 0, \qquad S.\alpha\rho dUd\rho = 0\,;$$

integration gives,

$$\text{const.} = S.\alpha\rho Ud\rho = TV\alpha\rho SU(V\alpha\rho\,.\,d\rho);$$

the perpendicular distance of a point on a geodetic line from the axis of revolution varies inversely as the cosine of the angle under which the geodetic crosses a parallel (or circle) on the surface.

(*f*) The differential equation, $S.\alpha\rho dUd\rho = 0$, is satisfied not only by the geodetics, but also by the circles, on a surface of revolution; give the explanation of this fact of calculation, and shew that it arises from the coincidence between the normal plane to the circle and the plane of the meridian of the surface.

(*g*) For any arbitrary surface, the equation of the geodetic may be thus transformed, $S.\nu d\rho d^2\rho = 0$; deduce this form, and shew that it expresses the normal property of the osculating plane.

(*h*) If the element of the geodetic be constant, $dTd\rho = 0$, then the general equation formerly assigned may be reduced to $V.\nu d^2\rho = 0$.

Under the same condition, $d^2\rho = -\nu^{-1}Sd\nu d\rho$.

(*i*) If the equation of a central surface of the second order be put under the form $f\rho = 1$, where the function $f$ is scalar, and homogeneous of the second dimension, then the differential of that function is of the form $df\rho = 2S.\nu d\rho$, where the normal vector, $\nu = \phi\rho$, is a distributive function of $\rho$ (homogeneous of the first dimension), $d\nu = d\phi\rho = \phi d\rho$.

This normal vector $\nu$ may be called the *vector of proximity* (namely, of the element of the surface to the centre); because its reciprocal, $\nu^{-1}$, represents in length and in direction the perpendicular let fall from the centre on the tangent plane to the surface.

(*k*) If we make $S\sigma\phi\rho = f(\sigma, \rho)$ *this* function $f$ is commutative with respect to the *two* vectors on which it depends,

$f(\rho, \sigma) = f(\sigma, \rho)$; it is also connected with the *former* function $f$, of a *single* vector $\rho$, by the relation, $f(\rho, \rho) = f\rho$: so that

$$f\rho = S\rho\phi\rho.$$

$fd\rho = Sd\rho d\nu$; $dfd\rho = 2S \,.\, d\nu d^2\rho$; for a geodetic, with constant element,

$$\frac{dfd\rho}{2fd\rho} + S\frac{d\nu}{\nu} = 0;$$

this equation is immediately integrable, and gives const. $= T\nu\surd(fUd\rho) =$ reciprocal of Joachimstal's product, $PD$.

($l$) If we give the name of "Didonia" to the curve (discussed by Delaunay) which, on a given surface and with a given perimeter, contains the greatest area, then for such a Didonian curve we have by quaternions the formula,

$$\int S \,.\, U\nu d\rho\delta\rho + c\delta\int Td\rho = 0,$$

where $c$ is an arbitrary constant.

Derive hence the differential equation of the second order, equivalent (through the constant $c$) to one of the third order,

$$c^{-1}d\rho = V \,.\, U\nu dUd\rho.$$

Geodetics are, therefore, that limiting case of Didonias for which the constant $c$ is infinite.

On a plane, the Didonia is a circle, of which the equation, obtained by integration from the general form, is

$$\rho = \varpi + cU\nu d\rho,$$

$\varpi$ being vector of centre, and $c$ being radius of circle.

($m$) Operating by $S \,.\, Ud\rho$, the general differential equation of the Didonia takes easily the following forms:

$$c^{-1}Td\rho = S\,(U\nu d\rho \,.\, dUd\rho);$$

$$c^{-1}Td\rho^2 = S\,(U\nu d\rho \,.\, d^2\rho);$$

$$c^{-1}Td\rho^3 = S \,.\, U\nu d\rho d^2\rho;$$

$$c^{-1} = S\frac{d^2\rho d\rho^{-2}}{U\nu d\rho}.$$

($n$) The vector $\omega$, of the centre of the osculating circle to a curve in space, of which the element $Td\rho$ is constant, has for expression,

$$\omega = \rho + \frac{d\rho^2}{d^2\rho}.$$

Hence for the general Didonia,

$$c^{-1} = S\frac{(\omega - \rho)^{-1}}{Uvd\rho};$$

$$T(\rho - \omega) = cSU\frac{\rho - \omega}{\nu d\rho}.$$

(*o*) Hence, the radius of curvature of any one Didonia varies, in general, proportionally to the cosine of the inclination of the osculating plane of the curve to the tangent plane of the surface.

And hence, by Meusnier's theorem, the difference of the squares of the curvatures of curve and surface is constant; the curvature of the surface meaning here the reciprocal of the radius of the sphere which osculates in the direction of the element of the Didonia.

(*p*) In general, for any curve on any surface, if $\xi$ denote the vector of the intersection of the axis of the element (or the axis of the circle osculating to the curve) with the tangent plane to the surface, then

$$\xi = \rho + \frac{\nu d\rho^3}{S.\nu d\rho d^2\rho}.$$

Hence, for the general Didonia, with the same signification of the symbols,

$$\xi = \rho - cU\nu d\rho;$$

and the constant $c$ expresses the length of the interval $\rho - \xi$, intercepted on the tangent plane, between the point of the curve and the axis of the osculating circle.

(*q*) If, then, a sphere be described, which shall have its centre on the tangent plane, and shall contain the osculating circle, the radius of this sphere shall always be equal to $c$.

(*r*) The recent expression for $\xi$, combined with the first form of the general differential equation of the Didonia, gives

$$d\xi = -cVdU\nu Ud\rho; \qquad V\nu d\xi = 0.$$

(*s*) Hence, or from the geometrical signification of the constant $c$, the known property may be proved, that if a developable surface be circumscribed about the arbitrary surface, so as to touch it along a Didonia, and if this developable be then unfolded into a plane, the curve will at the same time be flattened (generally) into a circular arc, with radius $= c$.

24. Find the condition that the equation

$$S\rho\,(\phi+f)^{-1}\rho=-1$$

may give three real values of $f$ for any given value of $\rho$. If $f$ be a function of a scalar parameter $\xi$, shew how to find the form of this function in order that we may have

$$-\nabla^2\xi=\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}+\frac{d^2\xi}{dz^2}=0.$$

Prove that the following is the relation between $f$ and $\xi$,

$$c\xi=\int\frac{df}{\sqrt{(g_1+f)(g_2+f)(g_3+f)}}=\int\frac{df}{\sqrt{m_f}}$$

in the notation of § 159. (Tait, *Trans. R. S. E.* 1873.)

25. Shew, after Hamilton, that the proof of Dupin's theorem, that "each member of one of three series of orthogonal surfaces cuts each member of each of the other series along its lines of curvature," may be expressed in quaternion notation as follows:

If $\quad S\nu d\rho=0,\quad S\nu' d\rho=0,\quad S.\nu\nu' d\rho=0$

be integrable, and if

$$S\nu\nu'=0,\quad\text{then}\quad V\nu' d\rho=0,\quad\text{makes}\quad S.\nu\nu' d\nu=0.$$

Or, as follows,

If $\quad S\nu\nabla\nu=0,\quad S\nu'\nabla\nu'=0,\quad S\nu''\nabla\nu''=0,\quad$ and $\quad V.\nu\nu'\nu''=0,$

then

$$S.\nu''(S\nu'\nabla)\,\nu=0,$$

where

$$\nabla=i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}.$$

26. Shew that the equation

$$V\alpha\rho=\rho V\beta\rho$$

represents the line of intersection of a cylinder and cone, of the second order, which have $\beta$ as a common generating line.

27. Two spheres are described, with centres at $A$, $B$, where $\overline{OA}=\alpha$, $OB=\beta$, and radii $a$, $b$. Any line, $OPQ$, drawn from the origin, cuts them in $P$, $Q$ respectively. Shew that the equation of the locus of intersection of $AP$, $BQ$ has the form

$$V\{\alpha+aU(\rho-\alpha)\}\{\beta+bU(\rho-\beta)\}=0.$$

Shew that this involves $\quad S.\alpha\beta\rho=0,$

and therefore that the left side is a scalar multiple of $V.\alpha\beta$, so that the locus is a plane curve.

Also shew that in the particular case

$$V\alpha\beta = 0,$$

the locus is the surface formed by the revolution of a Cartesian oval about its axis.

28. Integrate the equations

$$Vd\rho\,\{(\rho-\alpha)^{-1}-(\rho+\alpha)^{-1}\}=0,$$

$$V.d\rho V\beta\,\{(\rho-\alpha)^{-1}-(\rho+\alpha)^{-1}\}=0.$$

Shew that each represents a series of circles in space. What is the common property of the circles of each series? [See § 140, (10), (11).]

29. Express the general equation of a knot of any kind, on an endless cord, in the form

$$\rho = \phi\,(s),$$

pointing out precisely the nature of the function $\phi$.

What are the conditions to which $\phi$ must be subject, when possible distortions of the knot are to be represented? What are the conditions that the string may be capable of being brought into a mere ring form?

30. Find the envelop of the planes of equilateral triangles whose vertices are situated in three given lines in space. What does it become when two, or all three, of these lines intersect?

31. Form the equation of the surface described by a circle, when two given points in its axis are constrained to move on given straight lines. Also when the constraining lines are two concentric circles in one plane.

# CHAPTER XI.

## KINEMATICS.

353. In the present Chapter it is not proposed to give a connected account of even the elements of so extensive a subject as that indicated by the Title. All that is contemplated is to treat a few branches of the subject in such a way as to shew the student how to apply the processes of Quaternions.

And, with a view to the next Chapter, the portions selected for treatment will be those of most direct interest in their physical applications.

### *A. Kinematics of a Point.*

354. When a point's vector, $\rho$, is a function of the time $t$, we have seen (§ 36) that its vector-velocity is expressed by $\frac{d\rho}{dt}$ or, in Newton's notation, by $\dot{\rho}$.

That is, if

$$\rho = \phi t$$

be the equation of an orbit, *containing* (as the reader may see) *not merely the form of the orbit, but the law of its description also,* then

$$\dot{\rho} = \dot{\phi}' t$$

gives at once the form of the *Hodograph* and the law of its description.

This shews immediately that the *vector-acceleration of a point's motion,*

$$\frac{d^2\rho}{dt^2} \text{ or } \ddot{\rho},$$

*is the vector-velocity in the hodograph.* Thus the fundamental properties of the hodograph are proved almost intuitively.

355. Changing the independent variable, we have

$$\dot{\rho} = \frac{d\rho}{ds}\frac{ds}{dt} = v\rho',$$

if we employ the dash, as before, to denote $\frac{d}{ds}$.

This merely shews, in another form, that $\dot{\rho}$ expresses the velocity in magnitude and direction. But a second differentiation gives

$$\ddot{\rho} = \dot{v}\rho' + v^2\rho''.$$

This shews that the vector-acceleration can be resolved into two components, the first, $\dot{v}\rho'$, being in the direction of motion and equal in magnitude to the acceleration of the speed, $\dot{v}$ or $\frac{dv}{dt}$; the second, $v^2\rho''$, being in the direction of the radius of absolute curvature, and having for its amount the square of the speed multiplied by the curvature.

[It is scarcely conceivable that this important fundamental proposition can be proved more elegantly than by the process just given.]

356. If the motion be in a plane curve, we may write the equation as follows, so as to introduce the usual polar coördinates, $r$ and $\theta$,

$$\rho = r\alpha^{2\theta/\pi}\beta,$$

where $\alpha$ is a unit-vector perpendicular to, $\beta$ a unit-vector in, the plane of the curve.

Here, of course, $r$ and $\theta$ may be considered as connected by one scalar equation; or better, each may be looked on as a function of $t$. By differentiation we get

$$\dot{\rho} = \dot{r}\alpha^{2\theta/\pi}\beta + r\dot{\theta}\alpha\alpha^{2\theta/\pi}\beta,$$

which shews at once that $\dot{r}$ is the velocity along, $r\dot{\theta}$ that perpendicular to, the radius vector. Again,

$$\ddot{\rho} = (\ddot{r} - r\dot{\theta}^2)\alpha^{2\theta/\pi}\beta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\alpha\alpha^{2\theta/\pi}\beta,$$

which gives, by inspection, the components of acceleration along, and perpendicular to, the radius vector.

357. For *uniform acceleration in a constant direction*, we have at once

$$\ddot{\rho} = \alpha.$$

Whence $$\dot{\rho} = \alpha t + \beta,$$
where $\beta$ is the vector-velocity at epoch. This shews that the hodograph is a straight line described uniformly.

Also $$\rho = \frac{\alpha t^2}{2} + \beta t,$$
no constant being added if the origin be assumed to be the position of the moving point at epoch.

Since the resolved parts of $\rho$, parallel to $\beta$ and $\alpha$, vary respectively as the first and second powers of $t$, the curve is evidently a parabola (§ 31 ($f$)).

But we may easily deduce from the equation the following result,
$$T(\rho + \tfrac{1}{2}\beta\alpha^{-1}\beta) = -SU\alpha\left(\rho + \frac{\beta^2}{2}\alpha^{-1}\right),$$
the equation of a paraboloid of revolution, whose axis is $\alpha$. Also
$$S \, . \, \alpha\beta\rho = 0,$$
and therefore the distance of any point in the path from the point $-\frac{1}{2}\beta\alpha^{-1}\beta$ is equal to its distance from the line whose equation is
$$\rho = -\frac{\beta^2}{2}\alpha^{-1} + x\alpha V\alpha\beta.$$

Thus we recognise the focus and directrix property. [The student should remark here how the distances of the point of projection (which may, of course, be *any* point of the path) from the focus and from the directrix are represented in magnitude *and direction* by the two similar but different expressions
$$-\tfrac{1}{2}\beta\alpha^{-1}\beta \text{ and } -\tfrac{1}{2}\beta^2\alpha^{-1},$$
or $$-\tfrac{1}{2}(\beta\alpha^{-1})\beta \text{ and } -\tfrac{1}{2}\beta(\beta\alpha^{-1}).$$
This is an excellent example of the non-commutative character of quaternion multiplication.]

358. That the moving point may reach a point $\gamma$ (where $\gamma$ is, of course, coplanar with $\alpha$ and $\beta$) we must have, for some real value of $t$,
$$\gamma = \frac{\alpha}{2}t^2 + \beta t.$$

Now suppose $T\beta$, the speed of projection, to be given $= v$, and, for shortness, write $\varpi$ for $U\beta$.

Then $$\gamma = \frac{\alpha}{2}t^2 + vt\varpi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Since $$T\varpi = 1,$$

we have $$\frac{T\alpha^2 t^4}{4} - (v^2 - S\alpha\gamma)\, t^2 + T\gamma^2 = 0.$$

The values of $t^2$ are *real* if

$$(v^2 - S\alpha\gamma)^2 - T\alpha^2 T\gamma^2$$

is positive. Now, as $T\alpha T\gamma$ is never less than $S\alpha\gamma$, this condition evidently requires that $v^2 - S\alpha\gamma$ also shall be positive. Hence, when they are real, both values of $t^2$ are *positive.* Thus we have *four* values of $t$ which satisfy the conditions, and it is easy to see that since, disregarding the signs, they are equal two and two, each pair refer to the same path, but *described in opposite directions* between the origin and the extremity of $\gamma$. There are therefore, if any, in general two parabolas which satisfy the conditions. The directions of projection are (of course) given by the corresponding values of $\varpi$. These, in turn, are obtained at once from ($a$) in the form

$$v\varpi = \frac{1}{t}\gamma - \frac{t}{2}\alpha,$$

where $t$ has one or other of the values previously found.

359. The envelop of all the trajectories possible, with a given speed, evidently corresponds to

$$(v^2 - S\alpha\gamma)^2 - T\alpha^2 T\gamma^2 = 0,$$

for then $\gamma$ is the vector of intersection of two indefinitely close paths in the same vertical plane.

Now $$v^2 - S\alpha\gamma = T\alpha T\gamma$$

is evidently the equation of a paraboloid of revolution of which the origin is the focus, the axis parallel to $\alpha$, and the directrix plane at a distance $\dfrac{v^2}{T\alpha}$.

All the ordinary problems connected with parabolic motion are easily solved by means of the above formulæ. Some, however, are even more easily treated by assuming a horizontal unit-vector in the plane of motion, and expressing $\beta$ in terms of it and of $\alpha$. But this must be left to the student.

360. For *acceleration directed to or from a fixed point*, we have, taking that point as origin, and putting $P$ for the magnitude of the central acceleration,

$$\ddot{\rho} = PU\rho.$$

From this, at once,

$$V\rho\ddot{\rho} = 0.$$

Integrating $\qquad V\rho\dot{\rho} = \gamma =$ a constant vector.

The interpretation of this simple formula is—*first*, $\rho$ and $\dot{\rho}$ are in a plane perpendicular to $\gamma$, hence the path is in a plane (of course passing through the origin); *second*, the doubled area of the triangle, two of whose sides are $\rho$ and $\dot{\rho}$ (that is, the moment of the velocity) is constant.

[It is scarcely possible to imagine that a more simple proof than this can be given of the fundamental facts, that a central orbit is a plane curve, and that equal areas are described by the radius vector in equal times.]

361. When the *law of acceleration to or from the origin is that of the inverse square of the distance*, we have

$$P = \frac{m}{T\rho^2},$$

where $m$ is *negative* if the acceleration be directed *to* the origin.

Hence $\qquad \ddot{\rho} = \dfrac{mU\rho}{T\rho^2}.$

The following beautiful method of integration is due to Hamilton. (See § 140, (2).)

Generally, $\qquad \dfrac{dU\rho}{dt} = -\dfrac{U\rho \, . \, V\rho\dot{\rho}}{T\rho^2} = -\dfrac{U\rho \, . \, \gamma}{T\rho^2},$

therefore $\qquad \ddot{\rho}\gamma = -m\dfrac{dU\rho}{dt},$

and $\qquad \dot{\rho}\gamma = \epsilon - mU\rho,$

where $\epsilon$ is a constant vector, perpendicular to $\gamma$, because

$$S\gamma\dot{\rho} = 0.$$

Hence, in this case, we have for the hodograph,

$$\dot{\rho} = \epsilon\gamma^{-1} - mU\rho \, . \, \gamma^{-1}.$$

Of the two parts of this expression, which are both vectors, the first is constant, and the second is constant in length. Hence the locus of the extremity of $\dot{\rho}$ is a circle in a plane perpendicular to $\gamma$ (i.e. parallel to the plane of the orbit), whose radius is $T \, . \, m\gamma^{-1}$, and whose centre is at the extremity of the vector $\epsilon\gamma^{-1}$.

[This equation contains the whole theory of the *Circular Hodograph*. Its consequences are developed at length in Hamilton's *Elements*.]

362. We may write the equations of this circle in the form

$$T(\dot{\rho} - \epsilon\gamma^{-1}) = T \, . \, m\gamma^{-1},$$

(a sphere), and

$$S\gamma\dot{\rho} = 0$$

(a plane through the origin, and through the centre of the sphere).

The equation of the orbit is found by operating by $V \, . \, \rho$ upon that of the hodograph. We thus obtain

$$\gamma = V \, . \, \rho\epsilon\gamma^{-1} + mT\rho\gamma^{-1},$$

or

$$\gamma^2 = S\epsilon\rho + mT\rho,$$

or

$$mT\rho = S\epsilon\,(\gamma^2\epsilon^{-1} - \rho)\,;$$

in which last form we at once recognise the focus and directrix property. This is in fact the equation of a conicoid of revolution about its principal axis ($\epsilon$), and the origin is one of the foci. The orbit is found by combining it with the equation of its plane,

$$S\gamma\rho = 0.$$

We see at once that $\gamma^2\epsilon^{-1}$ is the vector distance of the directrix from the focus; and similarly that the excentricity is $T \, . \, \epsilon m^{-1}$, and the major axis $2T\dfrac{m\gamma^2}{m^2 + \epsilon^2}$.

363. To take a simpler case: *let the acceleration vary as the distance from the origin.*

Then

$$\ddot{\rho} = \pm\, m^2\rho,$$

the upper or lower sign being used according as the acceleration is *from* or *to* the centre.

This is

$$\left(\frac{d^2}{dt^2} \mp m^2\right)\rho = 0.$$

Hence

$$\rho = \alpha\epsilon^{mt} + \beta\epsilon^{-mt}\,;$$

or

$$\rho = \alpha \cos mt + \beta \sin mt,$$

where $\alpha$ and $\beta$ are arbitrary, but constant, vectors; and $\epsilon$ is the base of Napier's logarithms.

The first is the equation of a hyperbola (§ 31, $k$) of which $\alpha$ and $\beta$ are the directions of the asymptotes; the second, that of an ellipse of which $\alpha$ and $\beta$ are semi-conjugate diameters.

Since

$$\dot{\rho} = m\,\{\alpha\epsilon^{mt} - \beta\epsilon^{-mt}\},$$

or

$$= m\,\{-\alpha \sin mt + \beta \cos mt\},$$

the hodograph is again a hyperbola or ellipse. But in the first case it is, if we neglect the change of dimensions indicated by the scalar factor $m$, conjugate to the orbit; in the case of the ellipse it is similar and similarly situated.

364. Again, *let the acceleration be as the inverse third power of the distance*, we have

$$\ddot{\rho} = \frac{mU\rho}{T\rho^3}.$$

Of course, we have, as usual,

$$V\rho\dot{\rho} = \gamma.$$

Also, operating by $S.\dot{\rho}$,

$$S\dot{\rho}\ddot{\rho} = \frac{mS\rho\dot{\rho}}{T\rho^4},$$

of which the integral is

$$\dot{\rho}^2 = C - \frac{m}{\rho^2},$$

the equation of energy.

Again, $$S\rho\ddot{\rho} = \frac{m}{\rho^2}.$$

Hence $$S\rho\ddot{\rho} + \dot{\rho}^2 = C,$$

or $$S\rho\dot{\rho} = Ct,$$

no constant being added if we reckon the time from the passage through the apse, where $S\rho\dot{\rho} = 0$.

We have, therefore, by a second integration,

$$\rho^2 = Ct^2 + C' \ldots\ldots\ldots\ldots\ldots(1).$$

[To determine $C'$, remark that

$$\rho\dot{\rho} = Ct + \gamma,$$

or $$\rho^2\dot{\rho}^2 = C^2t^2 - \gamma^2.$$

But $$\rho^2\dot{\rho}^2 = C\rho^2 - m \text{ (by the equation of energy)},$$
$$= C^2t^2 + CC' - m, \text{ by (1).}$$

Hence $$CC' = m - \gamma^2.]$$

To complete the solution, we have, by § 140 (2),

$$V\frac{\dot{\rho}}{\rho} = \frac{dU\rho}{dt}(U\rho)^{-1} = \frac{d}{dt}\log\frac{U\rho}{\beta},$$

where $\beta$ is a unit-vector in the plane of the orbit.

But $$V\frac{\dot{\rho}}{\rho} = -\frac{\gamma}{\rho^2}.$$

Hence $$\log\frac{U\rho}{\beta} = -\gamma\int\frac{dt}{Ct^2 + C'}.$$

The elimination of $t$ between this equation and (1) gives $T\rho$ in terms of $U\rho$, or the required equation of the path.

We may remark that if $\theta$ be the ordinary polar angle in the orbit,

$$\log \frac{U\rho}{\beta} = \theta U\gamma.$$

Hence we have and

$$\left.\begin{aligned} \theta &= -T\gamma \int \frac{dt}{Ct^2 + C'} \\ r^2 &= -(Ct^2 + C') \end{aligned}\right\},$$

from which the ordinary equations of Cotes' spirals can be at once found. [See Tait and Steele's *Dynamics of a Particle*, Appendix (A).]

365. *To find the conditions that a given curve may be the hodograph corresponding to a central orbit.*

If $\varpi$ be its vector, *given as a function of the time*, $\int\varpi dt$ is that of the orbit; hence the requisite conditions are given by

$$V\varpi\int\varpi dt = \gamma \quad\text{.........................(1)},$$

where $\gamma$ is a constant vector.

We may transform this into other shapes more resembling the Cartesian ones.

Thus $$V\dot{\varpi}\int\varpi dt = 0 \quad\text{.........................(2)},$$

and $$V\ddot{\varpi}\int\varpi dt + V\dot{\varpi}\varpi = 0.$$

From (2) $$\int\varpi dt = x\dot{\varpi},$$

and therefore by (1) $$xV\varpi\dot{\varpi} = \gamma,$$

or the curve is *plane*. And

$$xV\ddot{\varpi}\dot{\varpi} + V\dot{\varpi}\varpi = 0\,;$$

or eliminating $x$, $$\gamma V\dot{\varpi}\ddot{\varpi} = -(V\varpi\dot{\varpi})^2.$$

Now if $v'$ be the velocity in the hodograph, $R'$ its radius of curvature, $p'$ the perpendicular on the tangent; this equation gives at once

$$hv' = R'p'^2,$$

which agrees with known results.

366. *The equation of an epitrochoid or hypotrochoid*, referred to the centre of the fixed circle, is evidently

$$\rho = ai^{2\omega t/\pi}\alpha + bi^{2\omega_1 t/\pi}\alpha,$$

where $\alpha$ is a unit vector in the plane of the curve and $i$ another perpendicular to it. Here $\omega$ and $\omega_1$ are the angular velocities in the two circles, and $t$ is the time elapsed since the tracing point and the centres of the two circles were in one straight line.

Hence, for the length of an arc of such a curve,

$$s = \int T\dot{\rho}\,dt = \int dt\,\sqrt{\{\omega^2 a^2 + 2\omega\omega_1 ab\cos(\omega - \omega_1)\,t + \omega_1{}^2 b^2\}},$$

$$= \int dt\,\sqrt{\left\{(\omega a \mp \omega_1 b)^2 \pm 4\omega\omega_1 ab \begin{vmatrix}\cos^2\\ \sin^2\end{vmatrix} \frac{\omega - \omega_1}{2}\,t\right\}},$$

which is, of course, an elliptic function.

But when the curve becomes an epicycloid or a hypocycloid, $\omega a \mp \omega_1 b = 0$, and

$$s = 2\sqrt{(\pm\omega\omega_1 ab)}\int dt \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} \frac{\omega - \omega_1}{2}\,t,$$

which can be expressed in finite terms, as was first shewn by Newton.

The hodograph is another curve of the same class, whose equation is

$$\dot{\rho} = i\,(a\omega i^{2\omega t/\pi}\,\alpha + b\omega_1\,i^{2\omega_1 t/\pi}\,\alpha)\,;$$

and the acceleration is denoted in magnitude and direction by the vector

$$\ddot{\rho} = -\,a\omega^2 i^{2\omega t/\pi}\,\alpha - b\omega_1{}^2 i^{2\omega_1 t/\pi}\,\alpha.$$

Of course the equations of the common *Cycloid* and *Trochoid* may be easily deduced from these forms by making $a$ indefinitely great and $\omega$ indefinitely small, but the product $a\omega$ finite; and transferring the origin to the point

$$\rho = a\alpha.$$

## *B. Kinematics of a Rigid System.*

367. Let $i$ be the normal-vector to any plane.

Let $\varpi$ and $\rho$ be the vectors of any two points in a rigid plate in contact with the plane.

After any small displacement of the rigid plate in its plane, let $d\varpi$ and $d\rho$ be the increments of $\varpi$ and $\rho$.

Then $Sid\varpi = 0$, $Sid\rho = 0$; and, since $T(\varpi - \rho)$ is constant,

$$S(\varpi - \rho)(d\varpi - d\rho) = 0.$$

And we may evidently assume, consistently with these equations,

$$d\rho = \omega i\,(\rho - \tau),$$

$$d\varpi = \omega i\,(\varpi - \tau)\,;$$

where of course $\tau$ is the vector of some point in the plane, to a rotation $\omega$ about which the displacement is therefore equivalent.

Eliminating $\tau$, we have

$$\omega i = \frac{d(\varpi - \rho)}{\varpi - \rho},$$

which gives $\omega$, and thence $\tau$ is at once found.

For any other point $\sigma$ in the plane figure

$$Sid\sigma = 0,$$

$$S(\rho - \sigma)(d\rho - d\sigma) = 0. \quad \text{Hence } d\rho - d\sigma = \omega_1 i(\rho - \sigma).$$

$$S(\sigma - \varpi)(d\varpi - d\sigma) = 0. \quad \text{Hence } d\sigma - d\varpi = \omega_2 i(\sigma - \varpi).$$

From which, at once, $\omega_1 = \omega_2 = \omega$, and

$$d\sigma = \omega i(\sigma - \tau),$$

or this point also is displaced by a rotation $\omega$ about an axis through the extremity of $\tau$ and parallel to $i$.

368. In the case of a rigid body moving about a fixed point let $\varpi$, $\rho$, $\sigma$ denote the vectors of any three points of the body; the fixed point being origin.

Then $\varpi^2$, $\rho^2$, $\sigma^2$ are constant, and so are $S\varpi\rho$, $S\rho\sigma$, and $S\sigma\varpi$.

After any small displacement we have, for $\varpi$ and $\rho$,

$$\left.\begin{array}{r} S\varpi d\varpi = 0 \\ S\rho d\rho = 0 \\ S\varpi d\rho + S\rho d\varpi = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now these three equations are satisfied by

$$d\varpi = V\alpha\varpi, \quad d\rho = V\alpha\rho,$$

where $\alpha$ is *any* vector whatever. But if $d\varpi$ and $d\rho$ are *given*, then

$$Vd\varpi d\rho = V.V\alpha\varpi V\alpha\rho = \alpha S.\alpha\rho\varpi.$$

Operate by $S.V\varpi\rho$, and remember (1),

$$S^2\varpi d\rho = S^2\rho d\varpi = S^2.\alpha\rho\varpi.$$

Hence $$\alpha = \frac{Vd\varpi d\rho}{S\varpi d\rho} = \frac{Vd\rho d\varpi}{S\rho d\varpi} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now consider $\sigma$, $$\left.\begin{array}{l} S\sigma d\sigma = 0 \\ S\rho d\sigma = -S\sigma d\rho \\ S\varpi d\sigma = -S\sigma d\varpi \end{array}\right\},$$

$d\sigma = V\alpha\sigma$ satisfies them all, by (2), and we have thus the proposition that *any small displacement of a rigid body about a fixed point is equivalent to a rotation.*

369. *To represent the rotation of a rigid body about a given axis, through a given finite angle.* [This is a work of supererogation, if we consider the results of § 119. But it may be interesting to obtain these results in another manner.]

Let $\alpha$ be a unit-vector in the direction of the axis, $\rho$ the vector of any point in the body with reference to a fixed point in the axis, and $\theta$ the angle of rotation.

Then
$$\begin{aligned}\rho &= \alpha^{-1} S\alpha\rho + \alpha^{-1} V\alpha\rho,\\ &= -\alpha S\alpha\rho - \alpha V\alpha\rho.\end{aligned}$$

The rotation leaves, of course, the first part unaffected, but the second evidently becomes
$$-\alpha^{2\theta/\pi}\alpha V\alpha\rho,$$
or
$$-\alpha V\alpha\rho\cos\theta + V\alpha\rho\sin\theta.$$

Hence $\rho$ becomes
$$\begin{aligned}\rho_1 &= -\alpha S\alpha\rho - \alpha V\alpha\rho\cos\theta + V\alpha\rho\sin\theta,\\ &= (\cos\theta/2 + \alpha\sin\theta/2)\,\rho\,(\cos\theta/2 - \alpha\sin\theta/2),\\ &= \alpha^{\theta/\pi}\rho\alpha^{-\theta/\pi}.\end{aligned}$$

370. Hence *to compound two rotations about axes which meet,* we may evidently write, as the effect of an additional rotation $\phi$ about the unit vector $\beta$,
$$\rho_2 = \beta^{\phi/\pi}\rho_1\beta^{-\phi/\pi}.$$
Hence
$$\rho_2 = \beta^{\phi/\pi}\alpha^{\theta/\pi}\rho\alpha^{-\theta/\pi}\beta^{-\phi/\pi}.$$
If the $\beta$-rotation had been first, and then the $\alpha$-rotation, we should have had
$$\rho'_2 = \alpha^{\theta/\pi}\beta^{\phi/\pi}\rho\beta^{-\phi/\pi}\alpha^{-\theta/\pi},$$
and the non-commutative property of quaternion multiplication shews that we have *not*, in general,
$$\rho'_2 = \rho_2.$$

If $\alpha, \beta, \gamma$ be radii of the unit sphere to the corners of a spherical triangle whose angles are $\theta/2$, $\phi/2$, $\psi/2$, we know that
$$\gamma^{\psi/\pi}\beta^{\phi/\pi}\alpha^{\theta/\pi} = -1. \quad \text{(Hamilton, \textit{Lectures}, p. 267.)}$$
Hence
$$\beta^{\phi/\pi}\alpha^{\theta/\pi} = -\gamma^{-\psi/\pi},$$
and we may write
$$\rho_2 = \gamma^{-\psi/\pi}\rho\gamma^{\psi/\pi},$$
or, *successive rotations about radii to two corners of a spherical triangle, and through angles double of those of the triangle, are*

*equivalent to a single rotation about the radius to the third corner, and through an angle double of the exterior angle of the triangle.*

Thus any number of successive *finite* rotations of a system, of which one point is fixed, may be compounded into a single rotation about a definite axis.

371. When the rotations are indefinitely small, the effect of one is, by § 369,

$$\rho_1 = \rho + \mathfrak{a} V\alpha\rho,$$

and for the two, neglecting products of small quantities,

$$\rho_2 = \rho + \mathfrak{a} V\alpha\rho + \mathfrak{b} V\beta\rho,$$

$\mathfrak{a}$ and $\mathfrak{b}$ representing the angles of rotation about the unit-vectors $\alpha$ and $\beta$ respectively.

But this is equivalent to

$$\rho_2 = \rho + T(\mathfrak{a}\alpha + \mathfrak{b}\beta)\, VU(\mathfrak{a}\alpha + \mathfrak{b}\beta)\, \rho,$$

representing a rotation through an angle $T(\mathfrak{a}\alpha + \mathfrak{b}\beta)$, about the unit-vector $U(\mathfrak{a}\alpha + \mathfrak{b}\beta)$. Now the latter is the *direction*, and the former the *length*, of the diagonal of the parallelogram whose sides are $\mathfrak{a}\alpha$ and $\mathfrak{b}\beta$.

We may write these results more simply, by putting $\alpha$ for $\mathfrak{a}\alpha$, $\beta$ for $\mathfrak{b}\beta$, where $\alpha$ and $\beta$ are now no longer unit-vectors, but represent by their versors the *axes*, and by their tensors the *angles* (small), of rotation.

Thus

$$\begin{aligned}\rho_1 &= \rho + V\alpha\rho,\\ \rho_2 &= \rho + V\alpha\rho + V\beta\rho,\\ &= \rho + V(\alpha + \beta)\,\rho.\end{aligned}$$

372. *Given the instantaneous axis in terms of the time, it is required to find the single rotation which will bring the body from any initial position to its position at a given time.*

If $\alpha$ be the initial vector of *any* point of the body, $\varpi$ the value of the same at time $t$, and $q$ the required quaternion, we have by § 119

$$\varpi = q\alpha q^{-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to $t$, this gives

$$\begin{aligned}\dot{\varpi} &= \dot{q}\alpha q^{-1} - q\alpha q^{-1}\dot{q}q^{-1},\\ &= \dot{q}q^{-1} \,.\, q\alpha q^{-1} - q\alpha q^{-1} \,.\, \dot{q}q^{-1},\\ &= 2V.(V\dot{q}q^{-1} \,.\, q\alpha q^{-1}).\end{aligned}$$

But

$$\dot{\varpi} = V\epsilon\varpi = V.\epsilon q\alpha q^{-1}.$$

Hence, as $q\alpha q^{-1}$ may be any vector whatever in the displaced body, we must have

$$\epsilon = 2V\dot{q}q^{-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

This result may be stated in even a simpler form than (2), for we have always, whatever quaternion $q$ may be,

$$S\dot{q}q^{-1} = \frac{dTq}{dt}(Tq)^{-1},$$

and, therefore, if we suppose the tensor of $q$, which, as it is not involved in $q(\quad)q^{-1}$, may have any value whatever, to be a constant (unity, for instance), we may write (2) in the form

$$\epsilon q = 2\dot{q} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

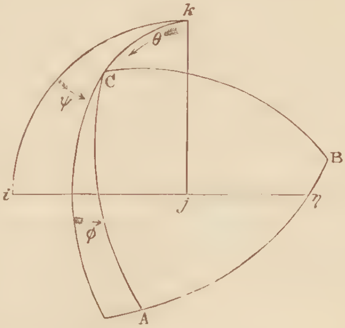
An immediate consequence, which will be of use to us later, is

$$q \cdot q^{-1}\epsilon q = 2\dot{q} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

373. *To express* q *in terms of the usual angles* $\psi$, $\theta$, $\phi$.

Here the vectors $i$, $j$, $k$ in the original position of the body correspond to $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, respectively, at time $t$. The transposition is defined to be effected by—*first*, a rotation $\psi$ about $k$; *second*, a rotation $\theta$ about the new position of the line originally coinciding with $j$; *third*, a rotation $\phi$ about the final position of the line at first coinciding with $k$.

This selection of angles, in terms of which the quaternion is to be expressed, is essentially unsymmetrical, and therefore the results cannot be expected to be simple.



The rotation $\psi$ about $k$ has the operator

$$k^{\psi/\pi}(\quad)k^{-\psi/\pi}.$$

This converts $j$ into $\eta$, where

$$\eta = k^{\psi/\pi}jk^{-\psi/\pi} = j\cos\psi - i\sin\psi.$$

The body next rotates about $\eta$ through an angle $\theta$. This has the operator

$$\eta^{\theta/\pi}(\quad)\eta^{-\theta/\pi}.$$

It converts $k$ into

$$\overline{OC} = \zeta = \eta^{\theta/\pi} k \eta^{-\theta/\pi} = (\cos\theta/2 + \eta\sin\theta/2)\, k\, (\cos\theta/2 - \eta\sin\theta/2)$$
$$= k\cos\theta + \sin\theta\,(i\cos\psi + j\sin\psi).$$

The body now turns through the angle $\phi$ about $\zeta$, the operator being

$$\zeta^{\phi/\pi}(\quad)\,\zeta^{-\phi/\pi}.$$

Hence, omitting a few reductions, which we leave as excellent practice for the reader, we find

$$q = \zeta^{\phi/\pi}\eta^{\theta/\pi}k^{\psi/\pi}$$
$$= (\cos\phi/2 + \zeta\sin\phi/2)(\cos\theta/2 + \eta\sin\theta/2)(\cos\psi/2 + k\sin\psi/2)$$
$$= \cos(\phi+\psi)/2\,.\,\cos\theta/2 + i\sin(\phi-\psi)/2\,.\,\sin\theta/2 +$$
$$j\cos(\phi-\psi)/2\,.\,\sin\theta/2 + k\sin(\phi+\psi)/2\,.\,\cos\theta/2,$$

which is, of course, essentially unsymmetrical.

374. *To find the usual equations connecting $\psi$, $\theta$, $\phi$ with the angular velocities about three rectangular axes fixed in the body.*

Having the value of $q$ in last section in terms of the three angles, it may be useful to employ it, in conjunction with equation (3) of § 372, partly as a verification of that equation. Of course, this is an exceedingly roundabout process, and does not in the least resemble the simple one which is immediately suggested by quaternions.

We have $$2\dot{q} = \epsilon q = \{\omega_1\overline{OA} + \omega_2\overline{OB} + \omega_3\overline{OC}\}q,$$

whence $$2q^{-1}\dot{q} = q^{-1}\{\omega_1\overline{OA} + \omega_2\overline{OB} + \omega_3\overline{OC}\}q,$$

or $$2\dot{q} = q(i\omega_1 + j\omega_2 + k\omega_3).$$

This breaks up into the four (equivalent to three independent) equations

$$2\frac{d}{dt}[\cos(\phi+\psi)/2\,.\,\cos\theta/2]$$
$$= -\omega_1\sin(\phi-\psi)/2\,.\,\sin\theta/2 - \omega_2\cos(\phi-\psi)/2\,.\,\sin\theta/2$$
$$-\omega_3\sin(\phi+\psi)/2\,.\,\cos\theta/2,$$

$$2\frac{d}{dt}[\sin(\phi-\psi)/2\,.\,\sin\theta/2]$$
$$= \omega_1\cos(\phi+\psi)/2\,.\,\cos\theta/2 - \omega_2\sin(\phi+\psi)/2\,.\,\cos\theta/2$$
$$+\omega_3\cos(\phi-\psi)/2\,.\,\sin\theta/2,$$

$$2\frac{d}{dt}[\cos(\phi-\psi)/2\,.\,\sin\theta/2]$$
$$=\quad \omega_1\sin(\phi+\psi)/2\,.\,\cos\theta/2+\omega_2\cos(\phi+\psi)/2\,.\,\cos\theta/2$$
$$-\,\omega_3\sin(\phi-\psi)/2\,.\,\sin\theta/2,$$

$$2\frac{d}{dt}[\sin(\phi+\psi)/2\,.\,\cos\theta/2]$$
$$=-\,\omega_1\cos(\phi-\psi)/2\,.\,\sin\theta/2+\omega_2\sin(\phi-\psi)/2\,.\,\sin\theta/2$$
$$+\,\omega_3\cos(\phi+\psi)/2\,.\,\cos\theta/2.$$

From the second and third, eliminate $\dot\phi-\dot\psi$, and we get by inspection

$$\cos\theta/2\,.\,\dot\theta=(\omega_1\sin\phi+\omega_2\cos\phi)\cos\theta/2,$$

or $$\dot\theta=\omega_1\sin\phi+\omega_2\cos\phi \quad\text{.......................(1).}$$

Similarly, by eliminating $\dot\theta$ between the same two equations,

$$\sin\theta/2\,.\,(\dot\phi-\dot\psi)=\omega_3\sin\theta/2+\omega_1\cos\phi\cos\theta/2-\omega_2\sin\phi\cos\theta/2.$$

And from the first and last of the group of four

$$\cos\theta/2\,.\,(\dot\phi+\dot\psi)=\omega_3\cos\theta/2-\omega_1\cos\phi\sin\theta/2+\omega_2\sin\phi\sin\theta/2.$$

These last two equations give

$$\dot\phi+\dot\psi\cos\theta=\omega_3 \quad\text{.....................(2).}$$
$$\dot\phi\cos\theta+\dot\psi=(-\,\omega_1\cos\phi+\omega_2\sin\phi)\sin\theta+\omega_3\cos\theta.$$

From the last two we have

$$\dot\psi\sin\theta=-\,\omega_1\cos\phi+\omega_2\sin\phi \quad\text{............(3).}$$

(1), (2), (3) are the forms in which the equations are usually given.

375. *To deduce expressions for the direction-cosines of a set of rectangular axes, in any position, in terms of rational functions of three quantities only.*

Let $\alpha$, $\beta$, $\gamma$ be unit-vectors in the directions of these axes. Let $q$ be, as in § 372, the requisite quaternion operator for turning the cöordinate axes into the position of this rectangular system. Then

$$q=w+xi+yj+zk,$$

where, as in § 372, we may write

$$1=w^2+x^2+y^2+z^2.$$

Then we have $$q^{-1}=w-xi-yj-zk,$$

and therefore

$$\alpha=qiq^{-1}=(wi-x-yk+zj)(w-xi-yj-zk)$$
$$=(w^2+x^2-y^2-z^2)\,i+2\,(wz+xy)\,j+2\,(xz-wy)\,k,$$

where the coefficients of $i$, $j$, $k$ are the direction-cosines of $\alpha$ as required. A similar process gives by inspection those of $\beta$ and $\gamma$.

As given by Cayley*, after Rodrigues, they have a slightly different and somewhat less simple form—to which, however, they are easily reduced by putting

$$w = x/\lambda = y/\mu = z/\nu = 1/\kappa^{\frac{1}{2}}.$$

The geometrical interpretation of either set is obvious from the nature of quaternions. For (taking Cayley's notation) if $\theta$ be the angle of rotation: $\cos f$, $\cos g$, $\cos h$, the direction-cosines of the axis, we have

$$q = w + xi + yj + zk = \cos\theta/2 + \sin\theta/2\,.\,(i\cos f + j\cos g + k\cos h),$$

so that
$$w = \cos\theta/2,$$
$$x = \sin\theta/2\,.\,\cos f,$$
$$y = \sin\theta/2\,.\,\cos g,$$
$$z = \sin\theta/2\,.\,\cos h.$$

From these we pass at once to Rodrigues' subsidiary formulae,

$$\kappa = 1/w^2 = \sec^2\theta/2,$$
$$\lambda = x/w = \tan\theta/2\,.\,\cos f,$$
$$\&c. = \&c.$$

### C. *Kinematics of a Deformable System.*

376. By the definition of *Homogeneous Strain*, it is evident that if we take any three (non-coplanar) unit-vectors $\alpha$, $\beta$, $\gamma$ in an unstrained mass, they become after the strain other vectors, not necessarily unit-vectors, $\alpha_1$, $\beta_1$, $\gamma_1$.

Hence any other *given* vector, which of course may be thus expressed,
$$\rho = x\alpha + y\beta + z\gamma,$$
becomes
$$\rho_1 = x\alpha_1 + y\beta_1 + z\gamma_1,$$
and is therefore known if $\alpha_1$, $\beta_1$, $\gamma_1$ be given.

More precisely

$$\rho S\,.\,\alpha\beta\gamma = \alpha S\,.\,\beta\gamma\rho + \beta S\,.\,\gamma\alpha\rho + \gamma S\,.\,\alpha\beta\rho$$

becomes

$$\rho_1 S\,.\,\alpha\beta\gamma = \phi\rho S\,.\,\alpha\beta\gamma = \alpha_1 S\,.\,\beta\gamma\rho + \beta_1 S\,.\,\gamma\alpha\rho + \gamma_1 S\,.\,\alpha\beta\rho.$$

Thus the properties of $\phi$, as in Chapter V., enable us to study with great simplicity homogeneous strains in a solid or liquid.

---

* *Camb. and Dub. Math. Journal.* Vol. i. (1846).

For instance, *to find a vector whose direction is unchanged by the strain*, is to solve the equation

$$V\rho\phi\rho = 0, \quad \text{or} \quad \phi\rho = g\rho \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

where $g$ is a scalar unknown.

[This vector equation is equivalent to *three* scalar equations, and contains only *three* unknown quantities; viz. *two* for the *direction* of $\rho$ (the *tensor* does not enter, or, rather, is a factor of each side), and the scalar $g$.]

We have seen that every such equation leads to a cubic in $g$ which may be written

$$g^3 - m_2 g^2 + m_1 g - m = 0,$$

where $m_2$, $m_1$, $m$ are scalars depending in a known manner on the constant vectors involved in $\phi$. This must have *one* real root, and may have *three*.

377. For simplicity let us assume that $\alpha, \beta, \gamma$ form a rectangular system, then we may operate on (1) by $S.\alpha$, $S.\beta$, and $S.\gamma$; and thus at once obtain the equation for $g$, in the form

$$\begin{vmatrix} S\alpha\alpha_1 + g, & S\alpha\beta_1, & S\alpha\gamma_1 \\ S\beta\alpha_1, & S\beta\beta_1 + g, & S\beta\gamma_1 \\ S\gamma\alpha_1, & S\gamma\beta_1, & S\gamma\gamma_1 + g \end{vmatrix} = 0.$$

To reduce this we have, for the term independent of $g$,

$$\begin{vmatrix} S\alpha\alpha_1, & S\alpha\beta_1, & S\alpha\gamma_1 \\ S\beta\alpha_1, & S\beta\beta_1, & S\beta\gamma_1 \\ S\gamma\alpha_1, & S\gamma\beta_1, & S\gamma\gamma_1 \end{vmatrix} = -S.\alpha\beta\gamma S.\alpha_1\beta_1\gamma_1 \text{ (Ex. 9, Chap. III.)},$$

which, if the mass be rigid, becomes $-1$.

The coefficient of the first power of $g$ is,

$$\Sigma(S\beta\beta_1 S\gamma\gamma_1 - S\beta\gamma_1 S\gamma\beta_1) = -\Sigma S.V\beta\gamma V\beta_1\gamma_1 = -\Sigma S.\alpha\beta_1\gamma_1.$$

Thus we can at once form the equation; which becomes, for the special case of a rigid system,

$$-1 - g(S\alpha\alpha_1 + S\beta\beta_1 + S\gamma\gamma_1) + g^2(S\alpha\alpha_1 + S\beta\beta_1 + S\gamma\gamma_1) + g^3 = 0,$$

or

$$(g-1)\{g^2 + g(1 + S\alpha\alpha_1 + S\beta\beta_1 + S\gamma\gamma_1) + 1\} = 0.$$

378. If we take $T\rho = C$ we consider a portion of the mass initially spherical. This becomes of course

$$T\phi^{-1}\rho_1 = C,$$

an ellipsoid, in the strained state of the body.

Or if we consider a portion which is spherical after the strain, i.e.

$$T\rho_1 = C,$$

its initial form was

$$T\phi\rho = C,$$

another ellipsoid. The relation between these ellipsoids is obvious from their equations. (See § 327.)

In either case the axes of the ellipsoid correspond to a rectangular set of three diameters of the sphere (§ 271). But we must carefully separate the cases in which these corresponding lines in the two surfaces are, and are not, coincident. For, in the former case there is *pure* strain, in the latter the strain is accompanied by rotation. Here we have at once the distinction pointed out by Stokes* and Helmholtz† between the cases of fluid motion in which there is, or is not, a velocity-potential. In ordinary fluid motion the distortion is of the nature of a pure strain, i.e. is differentially non-rotational; while in vortex motion it is essentially accompanied by rotation. But the resultant of two pure strains is generally a strain accompanied by rotation. The question before us beautifully illustrates the properties of the linear and vector function.

379. *To find the criterion of a pure strain.* Take $\alpha$, $\beta$, $\gamma$ now as unit-vectors parallel to the axes of the strain-ellipsoid, they become after the strain, $a\alpha$, $b\beta$, $c\gamma$, provided the strain be pure.

Hence $$\rho_1 = \phi\rho = -a\alpha S\alpha\rho - b\beta S\beta\rho - c\gamma S\gamma\rho.$$

And we have, for the criterion of a pure strain, the property of the function $\phi$, that it is *self-conjugate*, i.e.

$$S\rho\phi\sigma = S\sigma\phi\rho.$$

380. *Two pure strains, in succession, generally give a strain accompanied by rotation.* For if $\phi$, $\psi$ represent the strains, since they are pure we have

$$\left.\begin{array}{l} S\rho\phi\sigma = S\sigma\phi\rho \\ S\rho\psi\sigma = S\sigma\psi\rho \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But for the compound strain we have

$$\rho_1 = \chi\rho = \psi\phi\rho,$$

and we have *not* generally

$$S\rho\chi\sigma = S\sigma\chi\rho.$$

---

* *Cambridge Phil. Trans.* 1845.

† *Crelle*, vol. lv. 1857. See also *Phil. Mag.* (Supplement) June 1867.

For $$S\rho\psi\phi\sigma = S\sigma\phi\psi\rho,$$
by (1), and $\psi\phi$ is not generally the same as $\phi\psi$. (See Ex. 7 to Chapter V.)

*To find the lines which are most altered in length by the strain.*

Here $T\phi\rho$ is a maximum or minimum, while $T\rho$ is constant; so that
$$Sd\rho\phi'\phi\rho = 0, \quad S\rho d\rho = 0.$$
Hence $$\phi'\phi\rho = x\rho,$$
and the required lines are the principal vectors of $\phi'\phi$, which (§ 381) is obviously self-conjugate; i.e. denotes a pure strain.

381. The simplicity of this view of the question leads us to suppose that we may easily *separate the pure strain from the rotation in any case,* and exhibit the corresponding functions.

When the linear and vector function expressing a strain is self-conjugate the strain is pure. When not self-conjugate, it may be broken up into pure and rotational parts in various ways (analogous to the separation of a quaternion into the *sum* of a scalar and a vector part, or into the *product* of a tensor and a versor part), of which two are particularly noticeable. Denoting by a bar a self-conjugate function, we have thus either
$$\phi = \overline{\psi} + V.\epsilon(\ \ ),$$
$$\phi = q\overline{\varpi}(\ \ )q^{-1}, \quad \text{or} \quad \phi = \overline{\varpi}.q(\ \ )q^{-1},$$
where $\epsilon$ is a vector, and $q$ a quaternion (which may obviously be regarded as a mere versor). [The student must remark that, although the same letters have been employed (from habit) in writing the two last formulae, one is not a transformation of the other. In the first a pure strain is succeeded by a rotation, in the second the rotation is followed by the pure strain.]

That this is possible is seen from the fact that $\phi$ involves nine independent constants, while $\overline{\psi}$ and $\overline{\varpi}$ each involve six, and $\epsilon$ and $q$ each three. If $\phi'$ be the function conjugate to $\phi$, we have
$$\phi' = \overline{\psi} - V.\epsilon(\ \ ),$$
so that $$2\overline{\psi} = \phi + \phi',$$
and $$2V.\epsilon(\ \ ) = \phi - \phi',$$
which completely determine the first decomposition. This is, of course, perfectly well known in quaternions, but it does not seem to have been noticed as a theorem in the kinematics of strains that

there is always one, and but one, mode of resolving a strain into the geometrical composition of the separate effects of (1) a *pure* strain, and (2) a rotation accompanied by uniform dilatation perpendicular to its axis, the dilatation being measured by ($\sec.\ \theta - 1$) where $\theta$ is the angle of rotation.

In the second form (whose solution does not appear to have been attempted), we have

$$\phi = q\bar{\varpi}(\ \ )q^{-1},$$

where the pure strain precedes the rotation, and from this

$$\phi' = \bar{\varpi} . q^{-1}(\ \ )q,$$

or in the conjugate strain the rotation (reversed) is followed by the pure strain. From these

$$\begin{aligned}\phi'\phi &= \bar{\varpi} . q^{-1}\{q\bar{\varpi}(\ \ )q^{-1}\}q \\ &= \bar{\varpi}^2,\end{aligned}$$

and $\bar{\varpi}$ is to be found by the solution of a biquadratic equation*. It is evident, indeed, from the identical equation

$$S . \sigma\phi'\phi\rho = S . \rho\phi'\phi\sigma$$

that the operator $\phi'\phi$ is self-conjugate.

In the same way

$$\phi\phi'(\ \ ) = q\bar{\varpi}^2\{q^{-1}(\ \ )q\}q^{-1},$$

or $$q^{-1}(\phi\phi'\rho)q = \bar{\varpi}^2(q^{-1}\rho q) = \phi'\phi(q^{-1}\rho q),$$

---

* Suppose the cubic in $\bar{\varpi}$ to be

$$\bar{\varpi}^3 - g_2\bar{\varpi}^2 + g_1\bar{\varpi} - g = 0.$$

Now $\bar{\varpi}^2$ is equal to $\phi'\phi$, a known function, which we may call $\omega$. Thus

$$\bar{\varpi}^2 = \omega,$$

and therefore $\bar{\varpi}$ and $\omega$ are commutative in multiplication.

Eliminating $\bar{\varpi}$ between these equations we have, first,

$$(\bar{\varpi} - g_2)\omega + g_1\bar{\varpi} - g = 0 = \bar{\varpi}(\omega + g_1) - g_2\omega - g,$$

and finally $$\omega^3 + (2g_1 - g_2^2)\omega^2 + (g_1^2 - 2gg_2)\omega - g^2 = 0.$$

This must agree with the (known) cubic in $\omega$,

$$\omega^3 - m_2\omega^2 + m_1\omega - m = 0, \text{ suppose};$$

so that, by comparison of coefficients we have

$$2g_1 - g_2^2 = -m_2, \quad g_1^2 - 2gg_2 = m_1, \quad g^2 = m;$$

and thus $g$ is known, and $$g_2 = \frac{g_1^2 - m_1}{2\sqrt{m}},$$

where $$2g_1 - \frac{(g_1^2 - m_1)^2}{4m} = -m_2.$$

The values of the quantities $g$ being found, $\bar{\varpi}$ is given in terms of $\omega$ by the equation

$$\bar{\varpi} = \frac{g_2\omega + g}{\omega + g_1}.$$

which shew the relations between $\phi\phi'$, $\phi'\phi$, and $q$.

To determine $q$ we have

$$\phi\rho \,.\, q = q\varpi\rho$$

whatever be $\rho$, so that

$$S \,.\, Vq\,(\phi - \varpi)\,\rho = 0,$$

or

$$S \,.\, \rho\,(\phi' - \varpi)\, Vq = 0,$$

which gives

$$(\phi' - \varpi)\, Vq = 0.$$

The former equation gives evidently

$$Vq \parallel V \,.\, (\phi - \varpi)\,\alpha\,(\phi - \varpi)\,\beta$$

whatever be $\alpha$ and $\beta$; and the rest of the solution follows at once. A similar process gives us the solution when the rotation precedes the pure strain. [*Proc. R. S. E.* 1870—1.]

382. In general, if

$$\rho_1 = \phi\rho = -\alpha_1 S\alpha\rho - \beta_1 S\beta\rho - \gamma_1 S\gamma\rho,$$

the angle between any two lines, say $\rho$ and $\sigma$, becomes in the altered state of the body

$$\cos^{-1}(-S \,.\, U\phi\rho\, U\phi\sigma).$$

The plane $S\zeta\rho = 0$ becomes (with the notation of § 157)

$$S\phi'^{-1}\zeta\rho = S\zeta\phi^{-1}\rho = 0.$$

[For if $\lambda$, $\mu$ be any two vectors in it,

$$\zeta \parallel V\lambda\mu.$$

But they *become* $\phi\lambda$, $\phi\mu$, and the line perpendicular to both is

$$V\phi\lambda\phi\mu = m\phi'^{-1}V\lambda\mu.]$$

Hence the angle between the planes $S\zeta\rho = 0$, and $S\eta\rho = 0$, which is $\cos^{-1}(-S \,.\, U\zeta U\eta)$, becomes

$$\cos^{-1}(-S \,.\, U\phi'^{-1}\zeta U\phi'^{-1}\eta).$$

The *locus of lines equally elongated* is, of course,

$$T\phi U\rho = e,$$

or

$$T\phi\rho = eT\rho,$$

a cone of the second degree.

383. In the case of a *Simple Shear*, we have, obviously,

$$\rho_1 = \phi\rho = \rho + \beta S\alpha\rho,$$

where $\alpha$ is a unit vector, and

$$S\alpha\beta = 0.$$

The vectors which are unaltered in length are given by

$$T\rho_1 = T\rho,$$

or

$$2S\beta\rho S\alpha\rho + \beta^2 S^2\alpha\rho = 0,$$

which breaks up into $S.\alpha\rho = 0,$

and

$$S\rho(2\beta + \beta^2\alpha) = 0.$$

The intersection of this plane with the plane of $\alpha$, $\beta$ is perpendicular to $2\beta + \beta^2\alpha$. Let it be $\alpha + x\beta$, then

$$S.(2\beta + \beta^2\alpha)(\alpha + x\beta) = 0,$$

i.e.

$$2x - 1 = 0.$$

Hence the intersection required is

$$\alpha + \frac{\beta}{2}.$$

For the axes of the strain, one is of course $\alpha\beta$, and the others are found by making $T\phi U\rho$ a maximum and minimum.

Let

$$\rho = \alpha + x\beta,$$

then

$$\rho_1 = \phi\rho = \alpha + x\beta - \beta,$$

and

$$\frac{T\rho_1}{T\rho} = \text{max. or min.},$$

gives

$$x^2 - x + \frac{1}{\beta^2} = 0,$$

from which the values of $x$ (say $x_1$ and $x_2$) are found.

Also, as a verification, we must show that the lines of the body which become most altered in length are perpendicular to one another both before and after the shear.

Thus

$$S.(\alpha + x_1\beta)(\alpha + x_2\beta) = -1 + \beta^2 x_1 x_2,$$

should be $= 0$. It is so, since, by the equation in $x$,

$$x_1 x_2 = \frac{1}{\beta^2}.$$

Again

$$S\{\alpha + (x_1 - 1)\beta\}\{\alpha + (x_2 - 1)\beta\} = -1 + \beta^2\{x_1x_2 - (x_1 + x_2) + 1\},$$

ought also to be zero. And, in fact,

$$x_1 + x_2 = 1$$

by the equation for $x$; so that this also is verified.

384. We regret that our limits do not allow us to enter farther upon this very beautiful application. [The reader is referred to

Chapter X. of Kelland and Tait's *Introduction to Quaternions*; in which the treatment of linear and vector equations is *based* upon the theory of homogeneous strain; which, in its turn, is much more fully developed than in the present work.]

But it may be interesting to consider briefly the effects of *any* continuous displacements (of the particles of a body) by the help of the operator $\nabla$.

We have seen (§ 148) that the effect of the operator $-Sd\rho\nabla$, upon any scalar function of the vector of a point, is to produce total differentiation due to the passage from $\rho$ to $\rho + d\rho$.

Hence if $\sigma$ be the displacement of $\rho$, that of $\rho + \delta\rho$ is

$$\sigma - S(\delta\rho\nabla)\sigma.$$

Thus the strain of the group of particles near $\rho$ is such that

$$\phi\tau = \tau - S(\tau\nabla)\sigma.$$

[Here we virtually assume that $\sigma$ is a continuous function of $\rho$.]

But if this correspond to a linear dilatation $e$, combined with a rotation whose vector-axis is $\epsilon$, both being infinitesimal,

$$\phi\tau = \tau(1+e) + V\epsilon\tau.$$

Thus, for *all* values of $\tau$, each with its proper $e$,

$$V(e+\epsilon)\tau = -S(\tau\nabla)\sigma.$$

This gives at once (for instance by putting in succession for $\tau$ any three rectangular unit vectors)

$$2\epsilon - \Sigma e = \nabla\sigma,$$

from which we conclude as follows:—

If $\sigma$ (a *continuous* function of $\rho$) represent the vector displacement of a point situated at the extremity of the vector $\rho$ (drawn from the origin)

$S\nabla\sigma$ represents the consequent cubical compression of the group of points in the vicinity of that considered, and

$V\nabla\sigma$ represents twice the vector axis of rotation of the same group of points.

385. As an illustration, suppose we fix our attention upon a group of points which originally filled a small sphere about the extremity of $\rho$ as centre, whose equation referred to that point is

$$T\omega = c \text{ ..................................(1).}$$

After displacement $\rho$ becomes $\rho + \sigma$, and, by last section, $\rho + \omega$ becomes $\rho + \omega + \sigma - (S\omega\nabla)\sigma$. Hence the vector of the new surface which encloses the group of points (drawn from the extremity of $\rho+\sigma$) is

$$\omega_1=\omega-(S\omega\nabla)\,\sigma\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence $\omega$ is a homogeneous linear and vector function of $\omega_1$; or

$$\omega=\phi\omega_1,$$

and therefore, by (1), $$T\phi\omega_1=c,$$

the equation of the new surface, which is evidently a central surface of the second degree, and therefore, of course, an ellipsoid.

We may solve (2) with great ease by approximation, if we assume that $T\nabla\sigma$ is very small, and therefore that in the small term we may put $\omega_1$ for $\omega$; i.e. omit squares of small quantities; thus

$$\omega=\omega_1+(S\omega_1\nabla)\,\sigma.$$

386. *If the vector displacement of each point of a medium is in the direction of, and proportional to, the attraction exerted at that point by any system of material masses, the displacement is effected without rotation.*

For if $F\rho=C$ be the potential surface, we have $S\sigma d\rho$ a complete differential; and, by § 334,

$$V\nabla\sigma=0.$$

Conversely, *if there be no rotation, the displacements are in the direction of, and proportional to, the normal vectors to a series of surfaces.*

For $$0=-V.d\rho V\nabla\sigma=-(Sd\rho\nabla)\,\sigma+\nabla S\sigma d\rho,$$

where, in the last term, $\nabla$ acts on $\sigma$ alone.

Now, of the two terms on the right, the first is (§ 149, (4)) the complete differential $d\sigma$, and therefore the remaining term must be a complete differential. This, of course, means that

$$S\sigma d\rho$$

is a complete differential.

Thus, in a distorted system, there is no compression if

$$S\nabla\sigma=0,$$

and no rotation if $$V\nabla\sigma=0;$$

and evidently merely transference if $\sigma=\alpha=$ a constant vector, which is one case of

$$\nabla\sigma=0.$$

In the important case of $$\sigma=\nabla F\rho$$

there is (as proved already) no rotation, since

$$\nabla\sigma=\nabla^2F\rho$$

is evidently a scalar. In this case, then, there are only translation and compression, and the latter is at each point proportional to the density of a distribution of matter, which would give the potential $F\rho$. For if $r$ be such density, we have at once

$$\nabla^2 F\rho = 4\pi r^*.$$

### D. *Axes of Inertia.*

387. The *Moment of Inertia* of a body about a unit vector $\alpha$ as axis is evidently

$$Mh^2 = -\Sigma m (V\alpha\rho)^2,$$

where $\rho$ is the vector of the element $m$ of the mass, and the origin of $\rho$ is in the axis. [The letter $h$ has, for an obvious reason, been put here in place of the $k$ which is usually employed for the radius of gyration.]

Hence if we put $\beta = e^2\alpha/h$, where $e$ is constant, we have, as locus of the extremity of $\beta$,

$$Me^4 = -\Sigma m (V\beta\rho)^2 = -MS\beta\phi\beta \text{ (suppose)},$$

the well-known ellipsoid. The linear and vector function, $\phi$, depends only upon the distribution of matter about the (temporary) origin.

If $\varpi$ be the vector of the centre of inertia, $\sigma$ the vector of $m$ with respect to it, we have

$$\rho = \varpi + \sigma,$$

where (§ 31 (*e*)) $$\Sigma m\sigma = 0;$$

and therefore $$Mh^2 = -\Sigma m \{(V\alpha\varpi)^2 + (V\alpha\sigma)^2\}$$
$$= -M(V\alpha\varpi)^2 - MS\alpha\phi_1\alpha.$$

Here $\phi_1$ is the unique value of $\phi$ which corresponds to the distribution of matter relative to the centre of inertia. The equation last written gives the well-known relation between the moment of inertia about any line, and that about a parallel line through the centre of inertia.

Hence, to find the principal axes of inertia at any point (the origin, whose vector from the centre of inertia is $-\varpi$), note that $h$ is to be made max., min., or max.-min., with the condition

$$\alpha^2 = -1.$$

Thus we have $$S\alpha'(\varpi V\alpha\varpi + \phi_1\alpha) = 0,$$
$$S\alpha'\alpha = 0;$$

---

* *Proc. R. S. E.*, 1862—3.

therefore $\phi_1\alpha + \varpi V\alpha\varpi = p\alpha = h^2\alpha$ (by operating by $S.\alpha$).

Hence $$(\phi_1 - h^2 - \varpi^2)\alpha = -\varpi S\alpha\varpi \ldots\ldots\ldots\ldots(1),$$

determines the values of $\alpha$, $h^2$ being found from the cubic

$$S\varpi(\phi_1 - h^2 - \varpi^2)^{-1}\varpi = -1 \ldots\ldots\ldots\ldots(2).$$

Now the normal to

$$S\sigma(\phi_1 - h^2 - \varpi^2)^{-1}\sigma = -1 \ldots\ldots\ldots\ldots(3),$$

at the point $\sigma$ is parallel to

$$(\phi_1 - h^2 - \varpi^2)^{-1}\sigma.$$

But (3) passes through $-\varpi$, by (2), and *there* the normal is

$$(\phi_1 - h^2 - \varpi^2)^{-1}\varpi,$$

which, by (1), is parallel to one of the required values of $\alpha$. (3) is, of course (§ 288), one of the surfaces confocal with the ellipsoid

$$S.\sigma\phi_1^{-1}\sigma = -1.$$

Thus we prove Binet's theorem that *the principal axes at any point are normals to the three surfaces of the second degree, confocal with the central ellipsoid, which pass through that point.*

## EXAMPLES TO CHAPTER XI.

1. Form, from kinematical principles, the equation of the cycloid; and employ it to prove the well-known elementary properties of the arc, tangent, radius of curvature, and evolute, of the curve.

2. Interpret, kinematically, the equation

$$\dot{\rho} = aU(\beta t - \rho),$$

where $\beta$ is a given vector, and $a$ a given scalar.

Shew that it represents a plane curve; and give it in an integrated form independent of $t$.

3. If we write $$\varpi = \beta t - \rho,$$

the equation in (2) becomes

$$\beta - \dot{\varpi} = aU\varpi.$$

Interpret this kinematically; and find an integral.

What is the nature of the step we have taken in transforming from the equation of (2) to that of the present question?

4. The motion of a point in a plane being given, refer it to

(*a*) Fixed rectangular vectors in the plane.

(*b*) Rectangular vectors in the plane, revolving uniformly about a fixed point.

(*c*) Vectors, in the plane, revolving with different, but uniform, angular velocities.

(*d*) The vector radius of a fixed circle, drawn to the point of contact of a tangent from the moving point.

In each case translate the result into Cartesian coördinates.

5. Any point of a line of given length, whose extremities move in fixed lines in a given plane, describes an ellipse.

Shew how to find the centre, and axes, of this ellipse; and the angular velocity, about the centre of the ellipse, of the tracing point when the describing line rotates uniformly.

Transform this construction so as to shew that the ellipse is a hypotrochoid.

When the fixed lines are not in one plane, what is the locus?

6. A point, $A$, moves uniformly round one circular section of a cone; find the angular velocity of the point, $a$, in which the generating line passing through $A$ meets a subcontrary section, about the centre of that section.

7. Solve, generally, the problem of finding the path by which a point will pass in the least time from one given point to another, the speed at the point of space whose vector is $\rho$ being expressed by the given scalar function

$$f\rho.$$

Take also the following particular cases:—

(*a*) $f\rho = a$ while $S\alpha\rho > 1$,

$f\rho = b$ while $S\alpha\rho < 1$.

(*b*) $f\rho = TS\alpha\rho$.

(*c*) $f\rho = -\rho^2$. (Tait, *Trans. R. S. E.*, 1865.)

8. If, in the preceding question, $f\rho$ be such a function of $T\rho$ that any one swiftest path is a circle, every other such path is a circle, and all paths diverging from one point converge accurately in another. (Maxwell, *Camb. and Dub. Math. Journal*, IX. p. 9.)

9. Interpret, as results of the composition of successive conical rotations, the apparent truisms

$$\frac{\alpha}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=1,$$

and

$$\frac{\alpha}{\kappa}\frac{\kappa}{\iota}\frac{\iota}{\theta}\cdots\cdots\frac{\delta}{\gamma}\frac{\gamma}{\beta}\frac{\beta}{\alpha}=1.$$

(Hamilton, *Lectures*, p. 334.)

10. Interpret, in the same way, the quaternion operators

$$q=(\delta\epsilon^{-1})^{\frac{1}{2}}(\epsilon\zeta^{-1})^{\frac{1}{2}}(\zeta\delta^{-1})^{\frac{1}{2}},$$

and

$$q=\left(\frac{\alpha}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\epsilon}{\delta}\right)\left(\frac{\delta}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}. \quad (\textit{Ibid.})$$

11. Find the axis and angle of rotation by which one given rectangular set of unit-vectors $\alpha$, $\beta$, $\gamma$ is changed into another given set $\alpha_1$, $\beta_1$, $\gamma_1$.

12. Shew that, if $\phi\rho=\rho+V\epsilon\rho$,

the linear and vector operator $\phi$ denotes rotation about the vector $\epsilon$, together with uniform expansion in all directions perpendicular to it.

Prove this also by forming the operator which produces the expansion without the rotation, and that producing the rotation without the expansion; and finding their joint effect.

13. Express by quaternions the motion of a side of one right cone rolling uniformly upon another which is fixed, the vertices of the two being coincident.

14. Given the simultaneous angular velocities of a body about the principal axes through its centre of inertia, find the position of these axes in space at any assigned instant.

15. Find the linear and vector function, and also the quaternion operator, by which we may pass, in any simple crystal of the cubical system, from the normal to one given face to that to another. How can we use them to distinguish a series of faces belonging to the same zone?

16. Classify the simple forms of the cubical system by the properties of the linear and vector function, or of the quaternion operator, mentioned in (15) above.

17. Find the vector normal of a face which truncates symmetrically the edge formed by the intersection of two given faces.

18. Find the normals of a pair of faces symmetrically truncating the given edge.

19. Find the normal of a face which is equally inclined to three given faces.

20. Shew that the rhombic dodecahedron may be derived from the cube, or from the octahedron, by truncation of the edges.

21. Find the form whose faces replace, symmetrically, the edges of the rhombic dodecahedron.

22. Shew how the two kinds of hemihedral forms are indicated by the quaternion expressions.

23. Shew that the cube may be produced by truncating the edges of the regular tetrahedron. If an octahedron be cut from a cube, and cubes from its tetrahedra, all by truncation of edges, the two latter cubes coincide.

24. Point out the modifications in the auxiliary vector function required in passing to the pyramidal and prismatic systems respectively.

25. In the rhombohedral system the auxiliary quaternion operator assumes a singularly simple form. Give this form, and point out the results indicated by it.

26. Shew that if the hodograph be a circle, and the acceleration be directed to a fixed point; the orbit must be a conic section, which is limited to being a circle if the acceleration follow any other law than that of gravity.

27. In the hodograph corresponding to acceleration $f(D)$ directed towards a fixed centre, the curvature is inversely as $D^2 f(D)$.

28. If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut by any two common orthogonals, the sum of the two times of hodographically describing the two intercepted arcs (small or large) will be the same for the two hodographs. (Hamilton, *Elements*, p. 725.)

29. Employ the last theorem to prove, after Lambert, that the time of describing any arc of an elliptic orbit may be expressed in terms of the chord of the arc and the extreme radii vectores.

30. If $q\,(\;\;)\,q^{-1}$ be the operator which turns one set of rectangular unit-vectors $\alpha$, $\beta$, $\gamma$ into another set $\alpha_1$, $\beta_1$, $\gamma_1$, shew that there are three equations of the form

$$S\alpha\beta_1 - S\beta\alpha_1 = -\frac{4Sq\,S\gamma q}{Tq^2}.$$

31. If a ray, $\alpha$, fall on a fine, polished, wire $\gamma$, shew that on reflection it forms the surface

$$\rho^2 (S\alpha\gamma)^2 = \alpha^2 (S\gamma\rho)^2,$$

a right cone.

32. Find the path of a point, and the manner of its description, when

$$\dot{\rho} = (\rho - \alpha)^{-1} - (\rho + \alpha)^{-1}.$$

33. In the first problem of § 336 shew that

$$\nabla q^{-1} q = -\nabla v, \text{ or } \nabla\,.\,uq^{-1} = 0.$$

Also that $$(\nabla v)^2 = -2\nabla^2 v, \text{ or } 4u^{-\frac{1}{2}}\nabla^2 u^{\frac{1}{2}} = 0.$$

Again, shew that there are three equations of the form

$$\frac{dv}{dx}\nabla v = i\nabla^2 v + \nabla\frac{dv}{dx}.$$

From these last deduce, by a semi-Cartesian process, the result

$$u = \epsilon^v = C/T\rho^2,$$

as in the text.

34. Give the exact solution of

$$\omega_1 = \omega - S\omega\nabla\,.\,\sigma. \quad (\S\ 385.)$$

[Note that we may assume, $\sigma$ being given,

$$d\sigma = \phi d\rho,$$

where the constituents of $\phi$ are known functions of $\rho$. Thus we have what is wanted for the problem above :—viz.

$$-S\omega\nabla\,.\,\sigma = \phi\omega;$$

with various other important results, such as

$$\phi'\omega = -\nabla S\omega\sigma, \quad \nabla\sigma = \Sigma i\phi i, \text{ \&c.]}$$

# CHAPTER XII.

## PHYSICAL APPLICATIONS.

388. We propose to conclude the work by giving a few instances of the ready applicability of quaternions to questions of mathematical physics, upon which, even more than on the Geometrical or Kinematical applications, the real usefulness of the Calculus must mainly depend—except, of course, in the eyes of that section of mathematicians for whom Transversals and Anharmonic Pencils, &c. have a to us incomprehensible charm. Of course we cannot attempt to give examples in all branches of physics, nor even to carry very far our investigations in any one branch: this Chapter is not intended to teach Physics, but merely to shew by a few examples how expressly and naturally quaternions seem to be fitted for attacking the problems it presents.

We commence with a few general theorems in Dynamics—the formation of the equations of equilibrium and motion of a rigid system, some properties of the central axis, and the motion of a solid about its centre of inertia. The student may profitably compare, with the processes in the text, those adopted by Hamilton in his *Elements* (Book III., Chap. III., Section 8).

### *A. Statics of a Rigid System.*

389. When any forces act on a rigid body, the force $\beta$ at the point whose vector is $\alpha$, &c., then, if the body be slightly displaced, so that $\alpha$ becomes $\alpha + \delta\alpha$, the whole work done against the forces is

$$\Sigma S\beta\delta\alpha.$$

This must vanish if the forces are such as to maintain equilibrium. Hence *the condition of equilibrium of a rigid body is*

$$\Sigma S\beta\delta\alpha = 0.$$

For a displacement of translation $\delta\alpha$ is *any* constant vector, hence

$$\Sigma\beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

For a rotation-displacement, we have by § 371, $\epsilon$ being the axis, and $T\epsilon$ being indefinitely small,

$$\delta\alpha = V\epsilon\alpha,$$

and $$\Sigma S . \beta V\epsilon\alpha = \Sigma S . \epsilon V\alpha\beta = S . \epsilon\Sigma (V\alpha\beta) = 0,$$

*whatever* be $\epsilon$, hence $$\Sigma . V\alpha\beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

These equations, (1) and (2), are equivalent to the ordinary six equations of equilibrium.

390. In general, for any set of forces, let

$$\Sigma\beta = \beta_1,$$

$$\Sigma . V\alpha\beta = \alpha_1,$$

it is required *to find the points for which the couple* $\alpha_1$ *has its axis coincident with the resultant force* $\beta_1$. Let $\gamma$ be the vector of such a point.

Then for it the axis of the couple is

$$\Sigma . V(\alpha - \gamma)\beta = \alpha_1 - V\gamma\beta_1,$$

and by condition $$x\beta_1 = \alpha_1 - V\gamma\beta_1.$$

Operate by $S\beta_1$; therefore

$$x\beta_1^2 = S\alpha_1\beta_1,$$

and $$V\gamma\beta_1 = \alpha_1 - \beta_1^{-1} S\alpha_1\beta_1 = -\beta_1 V\alpha_1\beta_1^{-1},$$

or $$\gamma = V\alpha_1\beta_1^{-1} + y\beta_1,$$

a straight line (the *Central Axis*) parallel to the resultant force.

[If the resultant force and couple be replaced by an equivalent in the form of two forces, $\beta$ at $\alpha$, and $\beta'$ at $\alpha'$, we have

$$\beta + \beta' = \beta_1, \quad V\alpha\beta + V\alpha'\beta' = \alpha_1.$$

The volume of the tetrahedron whose opposite edges are $\beta$, $\beta'$ (acting as above stated) is as $S . \beta' V(\alpha - \alpha')\beta$. But

$$S . \beta'\alpha\beta = S\beta'\alpha_1, \quad S . \beta'\alpha'\beta = -S\beta\alpha_1,$$

so that the volume is as $S\alpha_1(\beta + \beta') = S\alpha_1\beta_1$, a constant whatever pair of equivalent forces be taken.]

391. *To find the points about which the couple is least.*

Here $$T(\alpha_1 - V\gamma\beta_1) = \text{minimum}.$$

Therefore $$S.(\alpha_1 - V\gamma\beta_1)V\beta_1\gamma' = 0,$$

where $\gamma'$ is any vector whatever. It is useless to try $\gamma' = \beta_1$, but we may put it in succession equal to $\alpha_1$ and to $V\alpha_1\beta_1$. Thus

$$S.\gamma V.\beta_1 V\alpha_1\beta_1 = 0,$$

and $$(V\alpha_1\beta_1)^2 - \beta_1^2 S.\gamma V\alpha_1\beta_1 = 0.$$

Hence $$\gamma = xV\alpha_1\beta_1 + y\beta_1,$$

and by operating with $S.V\alpha_1\beta_1$, we get

$$\frac{1}{\beta_1^2}(V\alpha_1\beta_1)^2 = x(V\alpha_1\beta_1)^2,$$

or $$\gamma = V\alpha_1\beta_1^{-1} + y\beta_1,$$

the same locus as in last section.

392. The couple vanishes if

$$\alpha_1 - V\gamma\beta_1 = 0.$$

This necessitates $$S\alpha_1\beta_1 = 0,$$

or the force must be *in* the plane of the couple. If this be the case,

$$\gamma = \alpha_1\beta_1^{-1} + x\beta_1,$$

still the central axis.

To assign the values of forces $\xi$, $\xi_1$, to act at $\epsilon$, $\epsilon_1$, and be equivalent to the given system.

$$\xi + \xi_1 = \beta_1,$$

$$V\epsilon\xi + V\epsilon_1\xi_1 = \alpha_1.$$

Hence $$V\epsilon\xi + V\epsilon_1(\beta_1 - \xi) = \alpha_1,$$

and $$\xi = (\epsilon - \epsilon_1)^{-1}(\alpha_1 - V\epsilon_1\beta_1) + x(\epsilon - \epsilon_1).$$

Similarly for $\xi_1$. The indefinite terms may be omitted, as they must evidently be equal and opposite. In fact they are any equal and opposite forces whatever acting in the line joining the given points.

393. If a system of parallel forces act on a rigid body, say

$$x\beta \text{ at } \alpha, \&c.$$

they have the single resultant $\beta\Sigma(x)$, at $\bar{\alpha}$, such that

$$\Sigma(x)V\beta\bar{\alpha} = V.\beta\Sigma(x\alpha).$$

Hence, whatever be the common direction of the forces, the resultant passes through

$$\bar{\alpha} = \frac{\Sigma (x\alpha)}{\Sigma (x)}.$$

If $\Sigma (x) = 0$, the resultant is simply the couple

$$V . \beta \Sigma (x\alpha).$$

By the help of these expressions for systems of parallel forces we can easily proceed to the case of forces generally.

Thus if any system of forces, $\beta$, act at points, $\alpha$, of a rigid body; and if $i, j, k$ be a system of rectangular unit vectors such that

$$bk = \Sigma (-kSk\beta) = \Sigma \beta,$$

the resultant force is

$$bk \text{ acting at } \frac{\Sigma (\alpha S\beta k)}{\Sigma (S\beta k)}, \text{ or } \frac{\phi k}{\Sigma (S\beta k)}$$

as we may write it. Take this as origin, then $\phi k = 0$.

The resultant couple, in the same way, is

$$V . i\Sigma (\alpha S\beta i) + V . j\Sigma (\alpha S\beta j),$$

or

$$V (i\phi i + j\phi j).$$

Now $\phi i$, $\phi j$, $\phi k$ are invariants, in the sense that they retain the same values however the forces and (with them) the system $i, j, k$ be made to rotate: provided they preserve their mutual inclinations, and the forces their points of application. For the $\alpha$s are constant, and quantities of the form $S\beta i$, $S\beta j$, or $S\beta k$ are not altered by the rotation.

We may select the positions of $i$ and $j$ so that $\phi i$ and $\phi j$ shall be perpendicular to one another. For this requires only

$$S . \phi i \phi j = 0, \text{ or } S . i\phi'\phi j = 0.$$

But (§ 381) $\phi'\phi$ is a self-conjugate function; and, by our change of origin, $k$ is parallel to one of its chief vectors. The desired result is secured if we take $i, j$ as the two others.

With these preliminaries we may easily prove Minding's Theorem :—

*If a system of forces, applied at given points of a rigid body, have their directions changed in any way consistent with the preservation of their mutual inclinations, they have in an infinite number of positions a single force as resultant. The lines of action of all such single forces intersect each of two curves fixed in space.*

The condition for the resultant's being a single force in the line whose vector is $\rho$ is

$$bVk\rho = V(i\phi i + j\phi j),$$

which may be written as

$$b\rho = xk - j\phi i + i\phi j.$$

That the two last terms, together, form a vector is seen by operating on the former equation by $S.k$; for we thus have

$$Sj\phi i - Si\phi j = 0.$$

We may write these equations for convenience as

$$b\rho = xk - j\alpha + i\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$Sj\alpha - Si\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

[The student must carefully observe that $\alpha$ and $\beta$ are now used in a sense totally different from that in which they first appeared, but for which they are no longer required. If this should puzzle him, he may change $\alpha$ into $\gamma$, and $\beta$ into $\delta$, in the last two equations and throughout the remainder of this section.]

Our object now must be to express $i$ and $j$ in terms of the single variable $k$, which is afterwards to be eliminated for the final result.

From (2) we find at once

$$\left.\begin{aligned} yi &= Vk\beta + kVk\alpha \\ yj &= -Vk\alpha + kVk\beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

whence we easily arrive at either of the following

$$\left.\begin{aligned} y &= Si\alpha + Sj\beta, \\ \text{or} \quad -y^2 &= \alpha^2 + \beta^2 + (Sk\alpha)^2 + (Sk\beta)^2 + 2S.k\alpha\beta \end{aligned}\right\} \quad \ldots\ldots(4).$$

Substituting for $i$ and $j$ in (1) their values (3), we have

$$\begin{aligned} -yb\rho &= -zk - \alpha S\alpha k - \beta S\beta k - \alpha\beta \\ &= (\varpi - z)k - \alpha\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5), \end{aligned}$$

where $\varpi$, which is now used for a linear and vector function, is defined by the equation

$$\varpi\rho = -\alpha S\alpha\rho - \beta S\beta\rho.$$

Obviously $\varpi(\alpha\beta) = 0,$

so that $(\varpi - z)^{-1}(\alpha\beta) = -\dfrac{1}{z}\alpha\beta.$

Thus $-yb(\varpi - z)^{-1}\rho = k + \dfrac{\alpha\beta}{z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$

Multiply together the respective members of (5) and (6), and take the scalar, and we have

$$y^2b^2S\rho(\varpi - z)^{-1}\rho = z - (S\alpha k)^2 - (S\beta k)^2 - 2Sk\alpha\beta + \frac{\alpha^2\beta^2}{z},$$

or, by (4),

$$= y^2 + z + \alpha^2 + \beta^2 + \frac{\alpha^2\beta^2}{z}$$

$$= y^2 + \frac{(\alpha^2 + z)(\beta^2 + z)}{z},$$

which, for $z = -\alpha^2$, or $z = -\beta^2$, gives as the required curves the focal conics of the system

$$S\rho(\varpi - z)^{-1}\rho = b^{-2}.$$

394. The preceding investigation was based on the properties of a system of parallel forces; and thus has a somewhat composite, semi-Cartesian, character.

That which follows is much more purely quaternionic. It is taken from the *Trans. R. S. E.* 1880.

When any number of forces act on a rigid system; $\beta_1$ at the point $\alpha_1$, $\beta_2$ at $\alpha_2$, &c., their resultant consists of the single force

$$\beta = \Sigma\beta$$

acting at the origin, and the couple

$$\kappa = -\Sigma V\beta\alpha \text{.........................} (1).$$

If these can be reduced to a single force, the equation of the line in which that force acts is evidently

$$V\bar{\beta}\rho = \Sigma V\beta\alpha \text{..............................} (2).$$

Now suppose the system of forces to turn about, preserving their magnitudes, their points of application, and their mutual inclinations, and let us find the fixed curves in space, each of which is intersected by the line (2) in every one of the infinite number of its positions.

Operating on (2) by $V.\bar{\beta}$, it becomes

$$\rho\bar{\beta}^2 - \bar{\beta}S\bar{\beta}\rho = \Sigma(\alpha S\beta\bar{\beta} - \beta S\alpha\bar{\beta}) = \phi\bar{\beta} - \phi'\bar{\beta}$$

with the notation of Chap. V. Now, however the forces may turn,

$$\phi\bar{\beta} = \Sigma\alpha S\beta\bar{\beta}$$

is an absolute constant; for each scalar factor as $S\beta_1\bar{\beta}$ is unaltered by rotation. Let us therefore change the origin, *i.e.* the value of each $\alpha$, so as to make

$$\Sigma\alpha S\beta\bar{\beta} = \phi\bar{\beta} = 0 \text{......................} (3).$$

This shews that $\beta$ is one of the three principal vectors of $\phi$, and we see in consequence that $\phi$ may be expressed in the form

$$\gamma' S\gamma(\quad) + \delta' S\delta(\quad);$$

where $\gamma$ and $\delta$ are unit vectors, forming with $\beta$ a rectangular system. They may obviously be so chosen that $\gamma'$ and $\delta'$ shall be at right angles to one another, but these (though constants) are not necessarily unit vectors.

Equation (2) is now

$$bV\beta\rho = V\gamma\gamma' + V\delta\delta' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2'),$$

where $b$ is the tensor, and $\beta$ the versor, of $\bar{\beta}$.

The condition that the force shall lie in the plane of the couple is, of course, included in this, and is found by operating by $S.\beta$. Thus

$$S(\delta\gamma' - \gamma\delta') = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

395. We have here all the data of the problem, and solutions can only differ from one another in the mode of attacking (2′) and (4).

Writing (4) in the form

$$S\gamma(\delta' + V\beta\gamma') = 0,$$

we have at once

whence

$$\left.\begin{aligned} t\gamma &= V\beta\delta' + \beta V\beta\gamma', \\ t\delta &= -V\beta\gamma' + \beta V\beta\delta' \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(4'),$$

where $t$ is an undetermined scalar.

By means of these we may put (2′) in the form

$$\begin{aligned} btV\beta\rho &= -V.\beta(V\gamma'\delta' + \gamma' S\gamma'\beta + \delta' S\delta'\beta) \\ &= -V.\beta(V\gamma'\delta' - \varpi\beta) \end{aligned}$$

where

$$\varpi = -\gamma' S\gamma'(\quad) - \delta' S\delta'(\quad).$$

Let the tensors of $\gamma'$ and $\delta'$ be $e_1$, $e_2$ respectively, and let $\beta'$ be a unit vector perpendicular to them, then we may write

$$bt\rho = x\beta - e_1 e_2 \beta' + \varpi\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Operating by $(\varpi + x)^{-1}$, and noting that

$$\varpi\beta' = 0,$$

we have

$$bt(\varpi + x)^{-1}\rho = \beta - \frac{e_1 e_2}{x}\beta' \ldots\ldots\ldots\ldots\ldots(5').$$

Taking the scalar of the product of (5) and (5′) we have

$$b^2 t^2 S\rho(\varpi + x)^{-1}\rho = \frac{1}{x}(x\beta - e_1 e_2 \beta')^2 + S\beta\varpi\beta.$$

But by (4′) we have

$$t^2 = S\beta\varpi\beta + e_1^2 + e_2^2 - 2e_1e_2S\beta\beta' \dots\dots\dots\dots\dots (6),$$

so that, finally,

$$b^2S\rho(\varpi + x)^{-1}\rho = 1 - \frac{(e_1^2 + x)(e_2^2 + x)}{xt^2} \dots\dots\dots (7).$$

Equation (7), in which $t^2$ is given by (6) in terms of $\beta$, is true for every point of every single resultant. But we get an immense simplification by assuming for $x$ either of the particular values $-e_1^2$ or $-e_2^2$. For then the right-hand side of (7) is reduced to unity, and the equation represents one or other of the focal conics of the system of confocal surfaces

$$S\rho(\varpi - h)^{-1}\rho = \frac{1}{b^2},$$

a point of each of which must therefore lie on the line (5).

396. A singular form, in which Minding's Theorem can be expressed, appears at once from equation (2′). For that equation is obviously the condition that the linear and vector function

$$-b\rho S\beta(\quad) + \gamma'S\gamma(\quad) + \delta'S\delta(\quad)$$

shall denote a pure strain.

Hence the following problem:—*Given a set of rectangular unit vectors, which may take any initial position: let two of them, after a homogeneous strain, become given vectors at right angles to one another, find what the third must become that the strain may be pure.* The locus of the extremity of the third is, for every initial position, one of the single resultants of Minding's system; and therefore passes through each of the fixed conics.

Thus we see another very remarkable analogy between strains and couples, which is in fact suggested at once by the general expression for the impure part of a linear and vector function.

397. The scalar $t$, which was introduced in equations (4′), is shewn by (6) to be a function of $\beta$ alone. In this connection it is interesting to study the surface of the fourth degree

$$S\tau\varpi\tau - (e_1^2 + e_2^2)\tau^2 - 2e_1e_2T\tau S\beta'\tau = 1,$$

where

$$\tau = \frac{1}{t}\beta.$$

But this may be left as an exercise.

Another form of $t$ (by 4′) is $S\gamma\gamma' + S\delta\delta'$.

Meanwhile (6) shews that for any assumed value of $\beta$ there are but two corresponding Minding lines. If, on the other hand, $\rho$ be given there are in general four values of $\beta$.

398. For variety, and with the view of further exploring this very interesting question, we may take a different mode of attacking equations (4) and (2′), which contain the whole matter. In what follows $b$ will be merged in $\rho$, so that the *scale* of the result will be altered.

Operating by $V.\beta$ we transform (2′) into

$$\rho + \beta S\beta\rho = -(\gamma S\gamma'\beta + \delta S\delta'\beta) \quad\ldots\ldots\ldots\ldots\ldots(2'').$$

Squaring both sides we have

$$\rho^2 + S^2\beta\rho = S\beta\varpi\beta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Since $\beta$ is a unit vector, this may be taken as the equation of a cyclic cone; and every central axis through the point $\rho$ lies upon it. For we have not yet taken account of (4), which is the condition that there shall be no couple.

To introduce (4), operate on (2″) by $S.\gamma'$ and by $S.\delta'$. We thus have, by a double employment of (4),

$$\left.\begin{aligned} S\gamma'\rho + S\gamma'\beta S\beta\rho &= S\gamma\varpi\beta \\ S\delta'\rho + S\delta'\beta S\beta\rho &= S\delta\varpi\beta \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Next, multiplying (8) by $S\beta\varpi\beta$, and adding to it the squares of (9), we have

$$\rho^2 S\beta\varpi\beta - 2S\beta\rho S\beta\varpi\rho - S\rho\varpi\rho = -S\beta\varpi^2\beta \quad\ldots\ldots\ldots(10).$$

This is a second cyclic cone, intersecting (8) in the four directions $\beta$. Of course it is obvious that (8) and (10) are unaltered by the substitution of $\rho + y\beta$ for $\rho$.

If we look on $\beta$ as given, while $\rho$ is to be found, (8) is the equation of a right cylinder, and (10) that of a central surface of the second degree.

399. A curious transformation of these equations may be made by assuming $\rho_1$ to be any other point on one of the Minding lines represented by (8) and (10). Introducing the factor $-\beta^2\,(=1)$ in the terms where $\beta$ does not appear, and then putting throughout

$$\beta \parallel \rho_1 - \rho \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

(8) becomes $\quad -\rho^2\rho_1^2 + S^2\rho\rho_1 = S(\rho_1 - \rho)\,\varpi\,(\rho_1 - \rho) \quad\ldots\ldots\ldots\ldots(8').$

As this is symmetrical in $\rho$, $\rho_1$, we should obtain only the same result by putting $\rho_1$ for $\rho$ in (8), and substituting again for $\beta$ as before.

From (10) we obtain the corresponding symmetrical result

$$(\rho^2 - S\rho\rho_1)\, S\rho_1\varpi\rho_1 + (\rho_1^2 - S\rho\rho_1)\, S\rho\varpi\rho = - S\rho\rho_1 S\,(\rho_1 - \rho)\,\varpi\,(\rho_1 - \rho) - S\,(\rho_1 - \rho)\,\varpi^2\,(\rho_1 - \rho) \ldots\ldots (10').$$

These equations become very much simplified if we assume $\rho$ and $\rho_1$ to lie respectively in any two conjugate planes; specially in the planes of the focal conics, so that $S\delta'\rho = 0$, and $S\gamma'\rho_1 = 0$.

For if the planes be conjugate we have

$$S\rho\varpi\rho_1 = 0,$$
$$S\rho\varpi^2\rho_1 = 0,$$

and if, besides, they be those of the focal conics,

$$S\rho\rho_1 = - S\beta'\rho S\beta'\rho_1,$$
$$S\rho\varpi^2\rho = e_1^2 S\rho\varpi\rho, \text{ \&c.},$$

and the equations are

$$-\rho^2\rho_1^2 + S^2\rho\rho_1 = S\rho_1\varpi\rho_1 + S\rho\varpi\rho, \ldots\ldots\ldots\ldots\ldots\ldots (8''),$$

and

$$\rho^2 S\rho_1\varpi\rho_1 + \rho_1^2 S\rho\varpi\rho = - S\rho_1\varpi^2\rho_1 - S\rho\varpi^2\rho \ldots\ldots\ldots (10'').$$

From these we have at once the equations of the two Minding curves in a variety of different ways. Thus, for instance, let

$$\rho_1 = p\delta'$$

and eliminate $p$ between the equations. We get the focal conic in the plane of $\beta'$, $\gamma'$. In this way we see that Minding lines pass through each point of each of the two curves; and by a similar process that every line joining two points, one on the one curve, the other on the other, is a Minding line.

400. Another process is more instructive. Note that, by the equations of condition above, we have

$$S^2\rho\rho_1 = \left(\frac{S\rho_1\varpi\rho_1}{e_2^2} - \rho_1^2\right)\left(\frac{S\rho\varpi\rho}{e_1^2} - \rho^2\right).$$

Then our equations become

$$\frac{S\rho\varpi\rho S\rho_1\varpi\rho_1}{e_1^2 e_2^2} - \frac{\rho_1^2 + e_1^2}{e_1^2} S\rho\varpi\rho - \frac{\rho^2 + e_2^2}{e_2^2} S\rho_1\varpi\rho_1 = 0,$$

and

$$(\rho^2 + e_2^2)\, S\rho_1\varpi\rho_1 + (\rho_1^2 + e_1^2)\, S\rho\varpi\rho = 0.$$

If we eliminate $\rho^2$ or $\rho_1^2$ from these equations, the resultant obviously becomes divisible by $S\rho\varpi\rho$ or $S\rho_1\varpi\rho_1$, and we at once obtain the equation of one of the focal conics.

401. In passing it may be well to notice that equation (10) may be written in the simpler form

$$S\,.\,\rho\beta\rho\varpi\beta + S\rho\varpi\rho = S\beta\varpi^2\beta.$$

Also it is easy to see that if we put

$$\theta = \rho S\beta\rho - (\varpi + \rho^2)\beta$$

we have (8) in the form $S\beta\theta = 0$,

and by the help of this (10) becomes

$$\theta^2 = S\rho\varpi\rho.$$

This gives another elegant mode of attacking the problem.

402. Another valuable transformation of (2″) is obtained by considering the linear and vector function, $\chi$ suppose, by which $\beta$, $\gamma$, $\delta$ are derived from the system $\beta'$, $U\gamma'$, $U\delta'$. For then we have obviously

$$\rho = x\chi\beta' + \chi\varpi^{\frac{1}{2}}\chi\beta' \text{.....................} (2''').$$

This represents any central axis, and the corresponding form of the Minding condition is

$$S\,.\,\gamma'\chi\varpi^{-\frac{1}{2}}\delta' = S\,.\,\delta'\chi\varpi^{-\frac{1}{2}}\gamma' \text{..................} (4'').$$

Most of the preceding formulae may be looked upon as results of the elimination of the function $\chi$ from these equations. This forms probably the most important feature of such investigations, so far at least as the quaternion calculus is concerned.

403. It is evident from (2‴) that the vector-perpendicular from the origin on the central axis parallel to $\chi\beta'$ is expressed by

$$\tau = \chi\varpi^{\frac{1}{2}}\chi\beta'.$$

But there is an infinite number of values of $\chi$ for which $U\tau$ is a given versor. Hence the problem;—to find the maximum and minimum values of $T\tau$, when $U\tau$ is given—*i.e. to find the surface bounding the region which is filled with the feet of perpendiculars on central axes.*

We have $$T\tau^2 = -S\,.\,\chi\beta'\varpi\chi\beta',$$
$$0 = T\tau S\,.\,\chi\beta' U\tau.$$

Hence $$0 = S\,.\,\dot{\chi}\beta'\varpi\chi\beta',$$
$$0 = S\,.\,\dot{\chi}\beta' U\tau.$$

But as $T\beta'$ is constant $$0 = S\,.\,\dot{\chi}\beta'\chi\beta'.$$

These three equations give at sight

$$(\varpi + u)\,\chi\beta' = u' U\tau,$$

where $u$, $u'$ are unknown scalars. Operate by $S\,.\,\chi\beta'$ and we have

$$-\,T^2\tau - u = 0,$$

so that $$S\tau\,(\varpi + \tau^2)^{-1}\,\tau = 0.$$

This differs from the equation of Fresnel's wave-surface only in having $\varpi + \tau^2$ instead of $\varpi + \tau^{-2}$, and denotes therefore the reciprocal of that surface. In the statical problem, however, we have $\varpi\beta' = 0$, and thus the corresponding wave-surface has zero for one of its parameters. [See § 435.]

[If this restriction be not imposed, the locus of the point

$$\tau = \chi\phi\chi\beta',$$

where $\phi$ is now any given linear and vector function whatever, will be found, by a process precisely similar to that just given, to be

$$S\,.\,(\tau - \phi'\beta')\,(\phi'\phi + \tau^2)^{-1}\,(\tau - \phi'\beta') = 0,$$

where $\phi'$ is the conjugate of $\phi$.]

### *B. Kinetics of a Rigid System.*

404. For the motion of a rigid system, we have of course

$$\Sigma S\,(m\ddot{\alpha} - \beta)\,\delta\alpha = 0,$$

by the general equation of Lagrange.

Suppose the displacements $\delta\alpha$ to correspond to a mere *translation*, then $\delta\alpha$ is *any* constant vector, hence

$$\Sigma\,(m\ddot{\alpha} - \beta) = 0,$$

or, if $\alpha_1$ be the vector of the centre of inertia, and therefore

$$\alpha_1 \Sigma m = \Sigma m\alpha,$$

we have at once $$\ddot{\alpha}_1 \Sigma m - \Sigma\beta = 0,$$

and the centre of inertia moves as if the whole mass were concentrated in it, and acted upon by all the applied forces.

405. Again, let the displacements $\delta\alpha$ correspond to a *rotation* about an axis $\epsilon$, passing through the origin, then

$$\delta\alpha = V\epsilon\alpha,$$

it being assumed that $T\epsilon$ is indefinitely small.

Hence $$\Sigma S \,.\, \epsilon V\alpha\,(m\ddot{\alpha} - \beta) = 0,$$
for *all* values of $\epsilon$, and therefore
$$\Sigma \,.\, V\alpha\,(m\ddot{\alpha} - \beta) = 0,$$
which contains the three remaining ordinary equations of motion.

Transfer the origin to the centre of inertia, i.e. put $\alpha = \alpha_1 + \varpi$, then our equation becomes
$$\Sigma V\,(\alpha_1 + \varpi)\,(m\ddot{\alpha}_1 + m\ddot{\varpi} - \beta) = 0\,;$$
or, since $\Sigma m\varpi = 0$,
$$\Sigma V\varpi\,(m\ddot{\varpi} - \beta) + V\alpha_1\,(\ddot{\alpha}_1\Sigma m - \Sigma\beta) = 0.$$
But (§ 404) $$\ddot{\alpha}_1\Sigma m - \Sigma\beta = 0,$$
hence our equation is simply
$$\Sigma V\varpi\,(m\ddot{\varpi} - \beta) = 0.$$

Now $\Sigma V\varpi\beta$ is the couple, about the centre of inertia, produced by the applied forces; call it $\zeta$, then
$$\Sigma m V\varpi\ddot{\varpi} = \zeta \text{..........................(1).}$$

406. Integrating once,
$$\Sigma m V\varpi\dot{\varpi} = \gamma + \int\zeta dt \text{ .....................(2).}$$

Again, as the motion considered is *relative* to the centre of inertia, it must be of the nature of rotation about some axis, in general variable. Let $\epsilon$ denote at once the direction of, and the angular velocity about, this axis. Then, evidently,
$$\dot{\varpi} = V\epsilon\varpi.$$

Hence, the last equation may be written
$$\Sigma m\varpi V\epsilon\varpi = \gamma + \int\zeta dt.$$
Operating by $S\,.\,\epsilon$, we get
$$\Sigma m\,(V\epsilon\varpi)^2 = S\epsilon\gamma + S\epsilon\int\zeta dt \text{ ..................(3).}$$

But, by operating directly by $2\int S\epsilon dt$ upon the equation (1), we get
$$\Sigma m\,(V\epsilon\varpi)^2 = -h^2 + 2\int S\epsilon\zeta dt \text{ ...............(4).}$$
Equations (2) and (4) contain the usual four integrals of the first order, $h$ being here an arbitrary constant, whose value depends upon the initial kinetic energy of the system. By § 387 we see how the principal moments of inertia are involved in the left member.

407. When no forces act on the body, we have $\zeta = 0$, and therefore

$$\Sigma m \varpi V \epsilon \varpi = \gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$\Sigma m \dot{\varpi}^2 = \Sigma m (V \epsilon \varpi)^2 = -h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and, from (5) and (6), $$S \epsilon \gamma = -h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

One interpretation of (6) is, that the kinetic energy of rotation remains unchanged: another is, that the vector $\epsilon$ terminates in an ellipsoid whose centre is the origin, and which therefore assigns the angular velocity when the direction of the axis is given; (7) shews that the extremity of the instantaneous axis is always in a plane fixed in space.

Also, by (5), (7) is the equation of the tangent plane to (6) at the extremity of the vector $\epsilon$. Hence the ellipsoid (6) *rolls* on the plane (7).

From (5) and (6), we have at once, as an equation which $\epsilon$ must satisfy,

$$\gamma^2 \Sigma . m (V \epsilon \varpi)^2 = -h^2 (\Sigma . m \varpi V \epsilon \varpi)^2.$$

This belongs to a cone of the second degree fixed in the body. Thus all the results of Poinsot regarding the motion of a rigid body under the action of no forces, the centre of inertia being fixed, are deduced almost intuitively: and the only difficulties to be met with in more complex properties of such motion are those of integration, which are inherent to the subject, and appear whatever analytical method is employed. (Hamilton, *Proc. R. I. A.* 1848.)

If we write (5) as

$$\phi^{-1} \epsilon = \gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

the special notation $\phi$ indicating that this linear and vector function is related to the principal axes of the body, and not to lines fixed in space, and consider the ellipsoid (of which $\epsilon$ is a semidiameter)

$$S \epsilon \phi^{-1} \epsilon = -h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

we may write the equation of a confocal ellipsoid (also fixed in the body) as

$$S \rho (\phi + p)^{-1} \rho = -h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Any tangent plane to this is

$$S . \sigma (\phi + p)^{-1} \rho = -h^2 \ldots\ldots\ldots\ldots\ldots\ldots(9).$$

If this plane be perpendicular to $\gamma$, we may write

$$(\phi + p)^{-1}\rho = x\gamma,$$

so that, by (5)

$$\rho = x(\epsilon + p\gamma) \qquad \text{.....................(10)}.$$

The plane (9) intercepts on $\gamma$ a quantity $h^2/xT\gamma$, which is constant by (8) and (10).

The vector velocity of the point $\rho$ is

$$V\epsilon\rho = pxV\epsilon\gamma = -pV\gamma\rho$$

(by two applications of (10)). Hence the point of contact, $\rho$, revolves about $\gamma$ with angular velocity $-pT\gamma$. That is, if the plane (9) be rough, and can turn about $\gamma$ as an axis, the ellipsoid (8) instead of sliding upon it, will make it rotate with uniform angular velocity. This is a very simple mode of obtaining one of Sylvester's remarkable results. (*Phil. Trans.* 1866.)

408. For a more formal treatment of the problem of the rotation of a rigid body, we may proceed as follows:—

Let $\alpha$ be the initial position of $\varpi$, $q$ the quaternion by which the body can be at one step transferred from its initial position to its position at time $t$. Then

$$\varpi = q\alpha q^{-1}$$

and Hamilton's equation (5) of last section becomes

$$\Sigma \,.\, mq\alpha q^{-1} V \,.\, \epsilon q\alpha q^{-1} = \gamma,$$

or

$$\Sigma \,.\, mq\,\{\alpha S \,.\, \alpha q^{-1}\epsilon q - q^{-1}\epsilon q\alpha^2\}\, q^{-1} = \gamma.$$

The vector $\gamma$ is now written for $\gamma + \int\zeta dt$ of § 406, as $\zeta$ is required for a new purpose. Thus $\gamma$ represents the resultant moment of momentum, and will be constant only if there is no applied couple.

Let

$$\phi\rho = \Sigma \,.\, m\,(\alpha S\alpha\rho - \alpha^2\rho) \qquad \text{...................(1)},$$

where $\phi$ (compare § 387) is a self-conjugate linear and vector function, whose constituent vectors are fixed in the body in its initial position. Then the previous equation may be written

$$q\phi\,(q^{-1}\epsilon q)\, q^{-1} = \gamma,$$

or

$$\phi\,(q^{-1}\epsilon q) = q^{-1}\gamma q.$$

For simplicity let us write

$$\left.\begin{array}{l} q^{-1}\epsilon q = \eta \\ q^{-1}\gamma q = \zeta \end{array}\right\} \qquad \text{..........................(2)}.$$

Then Hamilton's dynamical equation becomes simply

$$\phi\eta = \zeta \qquad \text{...... ..........................(3)}.$$

409. It is easy to see what the new vectors $\eta$ and $\zeta$ represent. For we may write (2) in the form

$$\left.\begin{array}{l}\epsilon = q\eta q^{-1}\\ \gamma = q\zeta q^{-1}\end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2');$$

from which it is obvious that $\eta$ is that vector in the initial position of the body which, at time $t$, becomes the instantaneous axis in the moving body. When no forces act, $\gamma$ is constant, and $\zeta$ is the initial position of the vector which, at time $t$, is perpendicular to the invariable plane.

410. The complete statement of the problem is contained in equations (2), (3) above, and (4) of § 372*. Writing them again, we have

$$q\eta = 2\dot{q}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\gamma q = q\zeta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\phi\eta = \zeta\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

We have only to eliminate $\zeta$ and $\eta$, and we get

$$2\dot{q} = q\phi^{-1}(q^{-1}\gamma q)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

in which $q$ is now the only unknown; $\gamma$, if variable, being supposed given in terms of $q$ and $t$.

It is hardly conceivable that any simpler, or more easily interpretable, expression for the motion of a rigid body can be presented until symbols are devised far more comprehensive in their meaning than any we yet have.

411. Before entering into considerations as to the integration of this equation, we may investigate some other consequences of the group of equations in § 410. Thus, for instance, differentiating (2), we have

$$\gamma\dot{q} + \dot{\gamma}q = \dot{q}\zeta + q\dot{\zeta},$$

and, eliminating $\dot{q}$ by means of (4),

$$\gamma q\eta + 2\dot{\gamma}q = q\eta\zeta + 2q\dot{\zeta},$$

---

* To these it is unnecessary to add

$$Tq = \text{constant},$$

as this constancy of $Tq$ is proved by the *form* of (4). For, had $Tq$ been variable, there must have been a quaternion in the place of the vector $\eta$. In fact,

$$\frac{d}{dt}(Tq)^2 = 2S\,.\,\dot{q}Kq = (Tq)^2 S\eta = 0.$$

whence, eliminating $\gamma$ by the help of (2),

$$\dot\zeta = V\zeta\eta + q^{-1}\dot\gamma q\,;$$

which gives, in the case when no forces act, the forms

$$\dot\zeta = V\zeta\phi^{-1}\zeta \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and

$$(\text{as } \dot\zeta = \phi\dot\eta)$$

$$\phi\dot\eta = -V.\eta\phi\eta \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

To each of these the term $q^{-1}\dot\gamma q$ must be added on the right, if forces act.

412. It is now desirable to examine the formation of the function $\phi$. By its definition § 408, (1), we have

$$\phi\rho = \Sigma\,.\,m\,(\alpha S\alpha\rho - \alpha^2\rho),$$

$$= -\Sigma\,.\,m\alpha V\alpha\rho.$$

Hence

$$-S\rho\phi\rho = \Sigma\,.\,m\,(TV\alpha\rho)^2,$$

so that $-S\rho\phi\rho$ is the moment of inertia of the body about the vector $\rho$, multiplied by the square of the tensor of $\rho$. Compare § 387. Thus the equation

$$S\rho\phi\rho = -h^2,$$

evidently belongs to an ellipsoid, of which the radii-vectores are inversely as the square roots of the moments of inertia about them; so that, if $i, j, k$ be taken as unit-vectors in the directions of its axes respectively, we have

$$\left.\begin{aligned} Si\phi i &= -A \\ Sj\phi j &= -B \\ Sk\phi k &= -C \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$A, B, C$, being the principal moments of inertia. Consequently

$$\phi\rho = -\{AiSi\rho + BjSj\rho + CkSk\rho\} \quad\ldots\ldots\ldots\ldots(9).$$

Thus the equation (7) for $\eta$ breaks up, if we put

$$\eta = i\omega_1 + j\omega_2 + k\omega_3,$$

into the three following scalar equations

$$\left.\begin{aligned} A\dot\omega_1 + (C-B)\,\omega_2\omega_3 &= 0 \\ B\dot\omega_2 + (A-C)\,\omega_3\omega_1 &= 0 \\ C\dot\omega_3 + (B-A)\,\omega_1\omega_2 &= 0 \end{aligned}\right\},$$

which are the same as those of Euler. Only, it is to be understood that the equations just written are not primarily to be considered as equations of rotation. They rather express, with reference to fixed axes in the initial position of the body, the motion of the extremity, $\omega_1$, $\omega_2$, $\omega_3$, of the vector corresponding to the instantaneous axis in the moving body. If, however, we consider $\omega_1$, $\omega_2$, $\omega_3$ as standing for their values in terms of $w$, $x$, $y$, $z$ (§ 416 below), or any other cöordinates employed to refer the body to fixed axes, they *are* the equations of motion.

Similar remarks apply to the equation which determines $\zeta$, for if we put

$$\zeta = i\varpi_1 + j\varpi_2 + k\varpi_3,$$

(6) may be reduced to three scalar equations of the form

$$\dot{\varpi}_1 - \left(\frac{1}{C} - \frac{1}{B}\right)\varpi_2\varpi_3 = 0.$$

413. Euler's equations in their usual form are easily deduced from what precedes. For, let

$$\boldsymbol{\phi}\rho = q\phi\,(q^{-1}\rho q)\,q^{-1}$$

whatever be $\rho$; that is, let $\boldsymbol{\phi}$ represent with reference to the moving principal axes what $\phi$ represents with reference to the principal axes in the initial position of the body, and we have

$$\begin{aligned}\boldsymbol{\phi}\dot{\epsilon} &= q\phi\,(q^{-1}\dot{\epsilon}q)\,q^{-1} = q\phi\dot{\eta}q^{-1}\\ &= q\dot{\zeta}q^{-1} \qquad\quad = qV(\zeta\phi^{-1}\zeta)\,q^{-1}\\ &= -qV(\eta\phi\eta)\,q^{-1}\\ &= -V.\,q\eta\phi\eta q^{-1}\\ &= -V.\,q\eta q^{-1}q\phi\,(q^{-1}\epsilon q)\,q^{-1}\\ &= -V.\,\epsilon\boldsymbol{\phi}\epsilon,\end{aligned}$$

which is the required expression.

But perhaps the simplest mode of obtaining this equation is to start with Hamilton's unintegrated equation, which for the case of no forces is simply

$$\Sigma\,.\,mV\varpi\ddot{\varpi} = 0.$$

But from $\qquad \dot{\varpi} = V\epsilon\varpi$

we deduce $\qquad \ddot{\varpi} = V\epsilon\dot{\varpi} + V\dot{\epsilon}\varpi$

$$= \varpi\epsilon^2 - \epsilon S\epsilon\varpi + V\dot{\epsilon}\varpi,$$

so that $\qquad \Sigma\,.\,m\,(V\epsilon\varpi S\epsilon\varpi - \dot{\epsilon}\varpi^2 + \varpi S\dot{\epsilon}\varpi) = 0.$

If we look at equation (1), and remember that $\boldsymbol{\phi}$ differs from $\phi$ simply in having $\varpi$ substituted for $\alpha$, we see that this may be written

$$V\epsilon\boldsymbol{\phi}\epsilon + \boldsymbol{\phi}\dot{\epsilon} = 0,$$

the equation before obtained. The first mode of arriving at it has been given because it leads to an interesting set of transformations, for which reason we append other two.

By (2)

$$\gamma = q\zeta q^{-1},$$

therefore $$0 = \dot{q}q^{-1} . q\zeta q^{-1} + q\dot{\zeta}q^{-1} - q\zeta q^{-1}\dot{q}q^{-1},$$

or $$q\dot{\zeta}q^{-1} = 2V. \gamma V\dot{q}q^{-1}$$
$$= V\gamma\epsilon.$$

But, by the beginning of this section, and by (5) of § 407, this is again the equation lately proved.

Perhaps, however, the following is neater. It occurs in Hamilton's *Elements*.

By (5) of § 407 $\phi\epsilon = \gamma$.

Hence $$\phi\dot{\epsilon} = -\dot{\phi}\epsilon = -\Sigma . m(\dot{\varpi}V\epsilon\varpi + \varpi V\epsilon\dot{\varpi})$$
$$= -\Sigma . m\dot{\varpi}S\epsilon\varpi$$
$$= -V. \epsilon\Sigma . m\varpi S\epsilon\varpi$$
$$= -V\epsilon\phi\epsilon.$$

414. However they are obtained, such equations as those of § 412 were shewn long ago by Euler to be integrable as follows.

Putting $$2\int\omega_1\omega_2\omega_3 dt = s,$$

we have $$A\omega_1^2 = A\Omega_1^2 + (B - C)s,$$

with other two equations of the same form. Hence

$$2dt = \frac{ds}{\left(\Omega_1^2 + \frac{B-C}{A}s\right)^{\frac{1}{2}}\left(\Omega_2^2 + \frac{C-A}{B}s\right)^{\frac{1}{2}}\left(\Omega_3^2 + \frac{A-B}{C}s\right)^{\frac{1}{2}}},$$

so that $t$ is known in terms of $s$ by an elliptic integral. Thus, finally, $\eta$ or $\zeta$ may be expressed in terms of $t$; and in some of the succeeding investigations for $q$ we shall suppose this to have been done. It is with this integration, or an equivalent one, that most writers on the farther development of the subject have commenced their investigations.

415. By § 406, $\gamma$ is evidently the vector moment of momentum of the rigid body; and the kinetic energy is

$$-\tfrac{1}{2}\Sigma . m\dot{\varpi}^2 = -\tfrac{1}{2}S\epsilon\gamma.$$

But $$S\epsilon\gamma = S . q^{-1}\epsilon q q^{-1}\gamma q = S\eta\zeta,$$

so that when no forces act

$$S\zeta\phi^{-1}\zeta = S\eta\phi\eta = -h^2.$$

But, by (2), we have also

$$T\zeta = T\gamma, \quad \text{or} \quad T\phi\eta = T\gamma,$$

so that we have, for the equations of the cones described in the initial position of the body by $\eta$ and $\zeta$, that is, for the cones described in the moving body by the instantaneous axis and by the perpendicular to the invariable plane,

$$h^2\zeta^2 + \gamma^2 S\zeta\phi^{-1}\zeta = 0,$$
$$h^2(\phi\eta)^2 + \gamma^2 S\eta\phi\eta = 0.$$

This is on the supposition that $\gamma$ and $h$ are constants. If forces act, these quantities are functions of $t$, and the equations of the cones then described in the body must be found by eliminating $t$ between the respective equations. The final results to which such a process will lead must, of course, depend entirely upon the way in which $t$ is involved in these equations, and therefore no general statement on the subject can be made.

416. Recurring to our equations for the determination of $q$, and taking first the case of no forces, we see that, if we assume $\eta$ to have been found (as in § 414) by means of elliptic integrals, we have to solve the equation

$$q\eta = 2\dot{q}^*,$$

---

* To get an idea of the nature of this equation, let us integrate it on the supposition that $\eta$ is a *constant* vector. By differentiation and substitution, we get

$$2\ddot{q} = \dot{q}\eta = \tfrac{1}{2}\eta^2 q.$$

Hence
$$q = Q_1 \cos\frac{T\eta}{2}t + Q_2 \sin\frac{T\eta}{2}t.$$

Substituting in the given equation we have

$$T\eta\left(-Q_1 \sin\frac{T\eta}{2}t + Q_2\cos\frac{T\eta}{2}t\right) = \left(Q_1\cos\frac{T\eta}{2}t + Q_2\sin\frac{T\eta}{2}t\right)\eta.$$

Hence
$$T\eta \,.\, Q_2 = Q_1\eta,$$
$$-T\eta \,.\, Q_1 = Q_2\eta,$$

which are virtually the same equation, and thus

$$q = Q_1\left(\cos\frac{T\eta}{2}t + U\eta\sin\frac{T\eta}{2}t\right)$$
$$= Q_1(U\eta)^{\frac{tT\eta}{\pi}}.$$

And the interpretation of $q(\ )q^{-1}$ will obviously then be a rotation about $\eta$ through the angle $tT\eta$, together with any other arbitrary rotation whatever. Thus any position whatever may be taken as the initial one of the body, and $Q_1(\ )Q_1^{-1}$ brings it to its required position at time $t=0$.

that is, we have to integrate a system of four other differential equations harder than the first.

Putting, as in § 412,

$$\eta = i\omega_1 + j\omega_2 + k\omega_3,$$

where $\omega_1$, $\omega_2$, $\omega_3$ are supposed to be known functions of $t$, and

$$q = w + ix + jy + kz,$$

this system is
$$\frac{1}{2}dt = \frac{dw}{W} = \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z},$$

where
$$\begin{aligned} W &= -\omega_1 x - \omega_2 y - \omega_3 z, \\ X &= \quad \omega_1 w + \omega_3 y - \omega_2 z, \\ Y &= \quad \omega_2 w + \omega_1 z - \omega_3 x, \\ Z &= \quad \omega_3 w + \omega_2 x - \omega_1 y; \end{aligned}$$

or, as suggested by Cayley to bring out the skew symmetry,

$$\begin{array}{rcccc} X = & \cdot & \omega_3 y & - \omega_2 z & + \omega_1 w, \\ Y = & - \omega_3 x & \cdot & + \omega_1 z & + \omega_2 w, \\ Z = & \omega_2 x & - \omega_1 y & \cdot & + \omega_3 w, \\ W = & - \omega_1 x & - \omega_2 y & - \omega_3 z & \cdot \ . \end{array}$$

Here, of course, one integral is

$$w^2 + x^2 + y^2 + z^2 = \text{constant}.$$

It may suffice thus to have alluded to a possible mode of solution, which, except for very simple values of $\eta$, involves very great difficulties. The quaternion solution, when $\eta$ is of constant length and revolves uniformly in a right cone, will be given later.

417. If, on the other hand, we eliminate $\eta$, we have to integrate

$$q\phi^{-1}(q^{-1}\gamma q) = 2\dot{q},$$

so that one integration theoretically suffices. But, in consequence of the present imperfect development of the quaternion calculus, the only known method of effecting this is to reduce the quaternion equation to a set of four ordinary differential equations of the first order. It may be interesting to form these equations.

Put
$$\begin{aligned} q &= w + ix + jy + kz, \\ \gamma &= ia + jb + kc, \end{aligned}$$

then, by ordinary quaternion multiplication, we easily reduce the given equation to the following set:

$$\frac{dt}{2}=\frac{dw}{W}=\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

where

$$\begin{array}{llll}
W=-x\mathfrak{A}-y\mathfrak{B}-z\mathfrak{C} & \text{or} & X= & .\quad y\mathfrak{C}-z\mathfrak{B}+w\mathfrak{A}, \\
X=\phantom{-}w\mathfrak{A}+y\mathfrak{C}-z\mathfrak{B} & & Y= & -x\mathfrak{C}\quad .\quad +z\mathfrak{A}+w\mathfrak{B}, \\
Y=\phantom{-}w\mathfrak{B}+z\mathfrak{A}-x\mathfrak{C} & & Z= & x\mathfrak{B}-y\mathfrak{A}\quad .\quad +w\mathfrak{C}, \\
Z=\phantom{-}w\mathfrak{C}+x\mathfrak{B}-y\mathfrak{A} & & W= & -x\mathfrak{A}-y\mathfrak{B}-z\mathfrak{C}\quad .\quad ,
\end{array}$$

and

$$\mathfrak{A}=\frac{1}{A}\left[a\left(w^2-x^2-y^2-z^2\right)+2x(ax+by+cz)+2w(bz-cy)\right],$$

$$\mathfrak{B}=\frac{1}{B}\left[b\left(w^2-x^2-y^2-z^2\right)+2y(ax+by+cz)+2w(cx-az)\right],$$

$$\mathfrak{C}=\frac{1}{C}\left[c\left(w^2-x^2-y^2-z^2\right)+2z(ax+by+cz)+2w(ay-bx)\right].$$

$W$, $X$, $Y$, $Z$ are thus *homogeneous* functions of $w$, $x$, $y$, $z$ of the third degree.

Perhaps the simplest way of obtaining these equations is to translate the group of § 410 into $w$, $x$, $y$, $z$ at once, instead of using the equation from which $\zeta$ and $\eta$ are eliminated.

We thus see that

$$\eta=i\mathfrak{A}+j\mathfrak{B}+k\mathfrak{C}.$$

One obvious integral of these equations ought to be

$$w^2+x^2+y^2+z^2=\text{constant},$$

which has been assumed all along. In fact, we see at once that

$$wW+xX+yY+zZ=0$$

identically, which leads to the above integral.

These equations appear to be worthy of attention, partly because of the homogeneity of the denominators $W$, $X$, $Y$, $Z$, but particularly as they afford (what does not appear to have been sought) the means of solving this celebrated problem *at one step*, that is, without the previous integration of Euler's equations (§ 412).

A set of equations identical with these, but not in a homogeneous form (being expressed, in fact, in terms of $\kappa$, $\lambda$, $\mu$, $\nu$ of § 375, instead of $w$, $x$, $y$, $z$), is given by Cayley (*Camb. and Dub.*

*Math. Journal*, vol. i. 1846), and completely integrated (in the sense of being reduced to quadratures) by assuming Euler's equations to have been previously integrated. (Compare § 416.)

Cayley's method may be even more easily applied to the above equations than to his own; and I therefore leave this part of the development to the reader, who will at once see (as in § 416) that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ correspond to $\omega_1$, $\omega_2$, $\omega_3$ of the $\eta$ type, § 412.

418. It may be well to notice, in connection with the formulae for direction cosines in § 375 above, that we may write

$$\mathfrak{A} = \frac{1}{A}\left[a\left(w^2+x^2-y^2-z^2\right)+2b\left(xy+wz\right)+2c\left(xz-wy\right)\right],$$

$$\mathfrak{B} = \frac{1}{B}\left[2a\left(xy-wz\right)+b\left(w^2-x^2+y^2-z^2\right)+2c\left(yz+wx\right)\right],$$

$$\mathfrak{C} = \frac{1}{C}\left[2a\left(xz+wy\right)+2b\left(yz-wx\right)+c\left(w^2-x^2-y^2+z^2\right)\right].$$

These expressions may be considerably simplified by the usual assumption, that one of the fixed unit-vectors ($i$ suppose) is perpendicular to the invariable plane, which amounts to assigning definitely the initial position of one line in the body; and which gives the relations

$$b=0, \quad c=0.$$

419. When forces act, $\gamma$ is variable, and the quantities $a$, $b$, $c$ will in general involve all the variables $w$, $x$, $y$, $z$, $t$, so that the equations of last section become much more complicated. The type, however, remains the same if $\gamma$ involves $t$ only; if it involve $q$ we must differentiate the equation, put in the form

$$\gamma = 2q\phi\left(q^{-1}\dot{q}\right)q^{-1},$$

and we thus easily obtain the differential equation of the second order

$$\psi = 4V.\dot{q}\phi\left(q^{-1}\dot{q}\right)q^{-1}+2q\phi\left(V.q^{-1}\ddot{q}\right)q^{-1};$$

if we recollect that, because $q^{-1}\dot{q}$ is a vector, we have

$$S.q^{-1}\ddot{q} = \left(q^{-1}\dot{q}\right)^2.$$

Though the above formula is remarkably simple, it must, in the present state of the development of quaternions, be looked on as intractable, except in certain very particular cases.

420. Another mode of attacking the problem, at first sight entirely different from that in § 408, but in reality identical with it, is to seek the linear and vector function which expresses the *Homogeneous Strain* which the body must undergo to pass from its initial position to its position at time $t$.

Let $$\varpi = \chi\alpha,$$

$\alpha$ being (as in § 408) the initial position of a vector of the body, $\varpi$ its position at time $t$. In this case $\chi$ is a linear and vector function. (See § 376.)

Then, obviously, we have, $\varpi_1$ being the vector of some other point, which had initially the value $\alpha_1$,

$$S\varpi\varpi_1 = S \, . \, \chi\alpha\chi\alpha_1 = S\alpha\alpha_1,$$

(a particular case of which is

$$T\varpi = T\chi\alpha = T\alpha)$$

and $$V\varpi\varpi_1 = V. \, \chi\alpha\chi\alpha_1 = \chi V\alpha\alpha_1.$$

These are necessary properties of the strain-function $\chi$, depending on the fact that in the present application the system is rigid.

421. The kinematical equation

$$\dot{\varpi} = V\epsilon\varpi$$

becomes $$\dot{\chi}\alpha = V. \, \epsilon\chi\alpha$$

(the function $\dot{\chi}$ being formed from $\chi$ by the differentiation of its constituents with respect to $t$).

Hamilton's kinetic equation

$$\Sigma \, . \, m\varpi V\epsilon\varpi = \gamma,$$

becomes $$\Sigma \, . \, m\chi\alpha V. \, \epsilon\chi\alpha = \gamma.$$

This may be written

$$\Sigma \, . \, m \, (\chi\alpha S \, . \, \epsilon\chi\alpha - \epsilon\alpha^2) = \gamma,$$

or $$\Sigma \, . \, m \, (\alpha S \, . \, \alpha\chi'\epsilon - \chi^{-1}\epsilon \, . \, \alpha^2) = \chi^{-1}\gamma,$$

where $\chi'$ is the conjugate of $\chi$.

But, because $$S \, . \, \chi\alpha\chi\alpha_1 = S\alpha\alpha_1,$$

we have $$S\alpha\alpha_1 = S \, . \, \alpha\chi'\chi\alpha_1,$$

whatever be $\alpha$ and $\alpha_1$, so that

$$\chi' = \chi^{-1}.$$

Hence $$\Sigma \, . \, m \, (\alpha S \, . \, \alpha\chi^{-1}\epsilon - \chi^{-1}\epsilon \, . \, \alpha^2) = \chi^{-1}\gamma,$$

or, by § 408 $$\phi\chi^{-1}\epsilon = \chi^{-1}\gamma.$$

422. Thus we have, as the analogues of the equations in §§ 408, 409,

$$\chi^{-1}\epsilon = \eta,$$

$$\chi^{-1}\gamma = \zeta,$$

and the former result $\dot{\chi}\alpha = V.\epsilon\chi\alpha$

becomes $\dot{\chi}\alpha = V.\chi\eta\chi\alpha = \chi V\eta\alpha.$

This is our equation to determine $\chi$, $\eta$ being supposed known. To find $\eta$ we may remark that

$$\phi\eta = \zeta,$$

and $\dot{\zeta} = \dot{\widehat{\chi^{-1}}}\gamma.$

But $\chi\chi^{-1}\alpha = \alpha,$

so that $\dot{\chi}\chi^{-1}\alpha + \chi\dot{\widehat{\chi^{-1}}}\alpha = 0.$

Hence $\dot{\zeta} = -\chi^{-1}\dot{\chi}\chi^{-1}\gamma$

$$= -V.\eta\chi^{-1}\gamma = V\zeta\eta = V.\zeta\phi^{-1}\zeta,$$

or $\phi\dot{\eta} = -V\eta\phi\eta.$

These are the equations we obtained before. Having found $\eta$ from the last we have to find $\chi$ from the condition

$$\chi^{-1}\dot{\chi}\alpha = V\eta\alpha.$$

423. We might, however, have eliminated $\eta$ so as to obtain an equation containing $\chi$ alone, and corresponding to that of § 410. For this purpose we have

$$\eta = \phi^{-1}\zeta = \phi^{-1}\chi^{-1}\gamma,$$

so that, finally, $\chi^{-1}\dot{\chi}\alpha = V.\phi^{-1}\chi^{-1}\gamma\alpha,$

or $\dot{\widehat{\chi^{-1}}}\alpha = V.\chi^{-1}\alpha\phi^{-1}\chi^{-1}\gamma,$

which may easily be formed from the preceding equation by putting $\chi^{-1}\alpha$ for $\alpha$, and attending to the value of $\dot{\widehat{\chi^{-1}}}$ given in last section.

424. We have given this process, though really a disguised form of that in §§ 408, 410, and though the final equations to which it leads are not quite so easily attacked in the way of integration as those there arrived at, mainly to shew how free a use we can make of symbolic functional operators in quaternions

without risk of error. It would be very interesting, however, to have the problem worked out afresh from this point of view by the help of the old analytical methods: as several new forms of long-known equations, and some useful transformations, would certainly be obtained.

425. As a verification, let us now try to pass from the final equation, in $\chi$ alone, of § 423 to that of § 410 in $q$ alone.

We have, obviously,

$$\varpi = q\alpha q^{-1} = \chi\alpha,$$

which gives the relation between $q$ and $\chi$.

[It shews, for instance, that, as

$$S.\beta\chi\alpha = S.\alpha\chi'\beta,$$

while $$S.\beta\chi\alpha = S.\beta q\alpha q^{-1} = S.\alpha q^{-1}\beta q,$$

we have $$\chi'\beta = q^{-1}\beta q,$$

and therefore that $$\chi\chi'\beta = q(q^{-1}\beta q)q^{-1} = \beta,$$

or $$\chi' = \chi^{-1}, \text{ as above.]}$$

Differentiating, we have

$$\dot{q}\alpha q^{-1} - q\alpha q^{-1}\dot{q}q^{-1} = \dot{\chi}\alpha.$$

Hence $$\begin{aligned}\chi^{-1}\dot{\chi}\alpha &= q^{-1}\dot{q}\alpha - \alpha q^{-1}\dot{q}\\ &= 2V.V(q^{-1}\dot{q})\alpha.\end{aligned}$$

Also $$\phi^{-1}\chi^{-1}\gamma = \phi^{-1}(q^{-1}\gamma q),$$

so that the equation of § 423 becomes

$$2V.V(q^{-1}\dot{q})\alpha = V.\phi^{-1}(q^{-1}\gamma q)\alpha,$$

or, as $\alpha$ may have any value whatever,

$$2V.q^{-1}q = \phi^{-1}(q^{-1}\gamma q),$$

which, if we put $$Tq = \text{constant}$$

as was originally assumed, may be written

$$2\dot{q} = q\phi^{-1}(q^{-1}\gamma q),$$

as in § 410.

### *C. Special Kinetic Problems.*

426. *To form the equation for Precession and Nutation.* Let $\sigma$ be the vector, from the centre of inertia of the earth, to a particle $m$ of its mass: and let $\rho$ be the vector of the disturbing body, whose mass is $M$. The vector-couple produced is evidently

$$M\Sigma \, . \, mV . \, \sigma \, \frac{U(\rho-\sigma)}{T^2(\rho-\sigma)}$$

$$= M\Sigma \, . \, m \, \frac{V\sigma\rho}{T^3(\rho-\sigma)}$$

$$= M\Sigma \, . \, \frac{mV\sigma\rho}{T^3\rho} \, \frac{1}{\left(1+\frac{2S\sigma\rho}{T^2\rho}+\frac{T^2\sigma}{T^2\rho}\right)^{\frac{3}{2}}}$$

$$= M\Sigma \, . \, \frac{mV\sigma\rho}{T^3\rho}\left(1-\frac{3S\sigma\rho}{T^2\rho}+\&c.\right),$$

no farther terms being necessary, since $\frac{T\sigma}{T\rho}$ is always small in the actual cases presented in nature. But, because $\sigma$ is measured from the centre of inertia,

$$\Sigma \, . \, m\sigma = 0.$$

Also, as in § 408, $\qquad \phi\rho = \Sigma \, . \, m(\sigma S\sigma\rho - \sigma^2\rho).$

Thus the vector-couple required is

$$\frac{3M}{T^5\rho} \, V . \rho\phi\rho.$$

Referred to coördinates moving with the body, $\phi$ becomes $\boldsymbol{\phi}$ as in § 413, and § 413 gives

$$\boldsymbol{\phi}\epsilon = \gamma = 3M\int\frac{V . \rho\boldsymbol{\phi}\rho}{T^5\rho} \, dt.$$

Simplifying the value of $\boldsymbol{\phi}$ by assuming that the earth has two principal axes of equal moment of inertia, we have

$$B\epsilon - (A-B)\,\alpha S\alpha\epsilon = \text{vector-constant} + 3M(A-B)\int\frac{V\alpha\rho S\alpha\rho}{T^5\rho} \, dt.$$

This gives $\qquad S\alpha\epsilon = \text{const.} = \Omega,$

whence $\qquad \epsilon = -\Omega\alpha + \alpha\dot{\alpha},$

so that, finally,

$$BV\alpha\ddot{\alpha} - A\Omega\dot{\alpha} = \frac{3M}{T^5\rho}(A-B)\,V\alpha\rho S\alpha\rho.$$

The most striking peculiarity of this equation is that the *form* of the solution is entirely changed, not modified as in ordinary cases of disturbed motion, according to the nature of the value of $\rho$.

Thus, when the right-hand side vanishes, we have an equation which, in the case of the earth, would represent the rolling of a cone fixed in the earth on one fixed in space, the angles of *both* being exceedingly small.

If $\rho$ be finite, but constant, we have a case nearly the same as that of a top, the axis on the whole revolving conically about $\rho$.

But if we assume the expression

$$\rho = r\,(j \cos mt + k \sin mt),$$

(which represents a circular orbit described with uniform speed,) $\alpha$ revolves on the whole conically about the vector $i$, perpendicular to the plane in which $\rho$ lies. (§§ 408—426, *Trans. R. S. E.*, 1868—9.)

427. *To form the equation of motion of a simple pendulum, taking account of the earth's rotation.* Let $\alpha$ be the vector (from the earth's centre) of the point of suspension, $\lambda$ its inclination to the plane of the equator, $a$ the earth's radius drawn to that point; and let the unit-vectors $i, j, k$ be fixed in space, so that $i$ is parallel to the earth's axis of rotation; then, if $\omega$ be the angular velocity of that rotation

$$\alpha = a\,[i \sin\lambda + (j \cos\omega t + k \sin\omega t)\cos\lambda] \ldots\ldots\ldots (1).$$

This gives
$$\dot{\alpha} = a\omega\,(-j \sin\omega t + k \cos\omega t)\cos\lambda$$
$$= \omega V i\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly
$$\ddot{\alpha} = \omega V i\dot{\alpha} = -\,\omega^2\,(\alpha - ai\sin\lambda) \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

428. Let $\rho$ be the vector of the bob $m$ referred to the point of suspension, $R$ the tension of the string, then if $\alpha_1$ be the direction of pure gravity

$$m\,(\ddot{\alpha} + \ddot{\rho}) = -\,mg\,U\alpha_1 - RU\rho \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which may be written

$$V\rho\ddot{\alpha} + V\rho\ddot{\rho} = \frac{g}{T\alpha_1}\,V\alpha_1\rho \ldots\ldots\ldots\ldots\ldots (5).$$

To this must be added, since $r$ (the length of the string) is constant,

$$T\rho = r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the equations of motion are complete.

429. These two equations (5) and (6) contain every possible case of the motion, from the most infinitesimal oscillations to the most rapid rotation about the point of suspension, so that it is necessary to adapt different processes for their solution in different cases. We take here only the ordinary Foucault case, to the degree of approximation usually given.

430. Here we neglect terms involving $\omega^2$. Thus we write

$$\ddot{\alpha} = 0,$$

and we write $\alpha$ for $\alpha_1$, as the difference depends upon the ellipticity of the earth. Also, attending to this, we have

$$\rho = -\frac{r}{a}\alpha + \varpi \text{ ........................ } (7),$$

where (by (6)) $$S\alpha\varpi = 0 \text{ ................................ } (8),$$

and terms of the order $\varpi^2$ are neglected.

With (7), (5) becomes

$$-\frac{r}{a} V\alpha\ddot{\varpi} = \frac{g}{a} V\alpha\varpi;$$

so that, if we write $$\frac{g}{r} = n^2 \text{ ............................... } (9),$$

we have $$V\alpha(\ddot{\varpi} + n^2\varpi) = 0 \text{ ...................... } (10).$$

Now, the two vectors $ai - \alpha \sin\lambda$ and $Vi\alpha$ have, as is easily seen, equal tensors; the first is parallel to the line drawn horizontally *northwards* from the point of suspension, the second horizontally *eastwards*.

Let, therefore, $$\varpi = x(ai - \alpha\sin\lambda) + yVi\alpha \text{ ............ } (11),$$

which ($x$ and $y$ being very small) is consistent with (6).

From this we have (employing (2) and (3), and omitting $\omega^2$)

$$\dot{\varpi} = \dot{x}(ai - \alpha\sin\lambda) + \dot{y}Vi\alpha - x\omega\sin\lambda\, Vi\alpha - y\omega(\alpha - ai\sin\lambda),$$

$$\ddot{\varpi} = \ddot{x}(ai - \alpha\sin\lambda) + \ddot{y}Vi\alpha - 2\dot{x}\omega\sin\lambda\, Vi\alpha - 2\dot{y}\omega(\alpha - ai\sin\lambda).$$

With this (10) becomes

$$V\alpha[\ddot{x}(ai - \alpha\sin\lambda) + \ddot{y}Vi\alpha - 2\dot{x}\omega\sin\lambda\, Vi\alpha - 2\dot{y}\omega(\alpha - ai\sin\lambda)$$
$$+ n^2x(ai - \alpha\sin\lambda) + n^2yVi\alpha] = 0,$$

or, if we note that $$V.\alpha Vi\alpha = a(ai - \alpha\sin\lambda),$$

$$(-\ddot{x} - 2\dot{y}\omega\sin\lambda - n^2x)\,aVi\alpha + (\ddot{y} - 2\dot{x}\omega\sin\lambda + n^2y)a(ai - \alpha\sin\lambda) = 0.$$

This gives at once $$\left.\begin{array}{l}\ddot{x} + n^2x + 2\omega\dot{y}\sin\lambda = 0\\ \ddot{y} + n^2y - 2\omega\dot{x}\sin\lambda = 0\end{array}\right\}\text{ .................. } (12),$$

which are the equations usually obtained; and of which the solution is as follows:—

If we transform to a set of axes revolving in the horizontal plane at the point of suspension, the direction of motion being from the positive (northward) axis of $x$ to the positive (eastward) axis of $y$, with angular velocity $\Omega$, so that

$$\left.\begin{aligned} x &= \xi\cos\Omega t - \eta\sin\Omega t \\ y &= \xi\sin\Omega t + \eta\cos\Omega t \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

and omit the terms in $\Omega^2$ and in $\omega\Omega$ (a process justified by the results, see equation (15)), we have

$$\left.\begin{aligned} (\ddot\xi + n^2\xi)\cos\Omega t - (\ddot\eta + n^2\eta)\sin\Omega t - 2\dot y(\Omega - \omega\sin\lambda) = 0 \\ (\ddot\xi + n^2\xi)\sin\Omega t + (\ddot\eta + n^2\eta)\cos\Omega t + 2\dot x(\Omega - \omega\sin\lambda) = 0 \end{aligned}\right\} \ldots(14).$$

So that, if we put $\Omega = \omega\sin\lambda$ ........................(15),

we have simply

$$\left.\begin{aligned} \ddot\xi + n^2\xi = 0 \\ \ddot\eta + n^2\eta = 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

the usual equations of elliptic motion about a centre of force in the centre of the ellipse. (*Proc. R. S. E.*, 1869.)

### *D. Geometrical and Physical Optics.*

431. *To construct a reflecting surface from which rays, emitted from a point, shall after reflection diverge uniformly, but horizontally.*

Using the ordinary property of a reflecting surface, we easily obtain the equation

$$S\,.\,d\rho\left(\frac{\beta + \alpha V\alpha\rho}{\rho}\right)^{\frac{1}{2}}\rho = 0.$$

By Hamilton's grand *Theory of Systems of Rays*, we at once write down the second form

$$T\rho - T(\beta + \alpha V\alpha\rho) = \text{constant}.$$

The connection between these is easily shewn thus. Let $\varpi$ and $\tau$ be any two vectors whose tensors are equal, then

$$\begin{aligned} \left(\frac{\tau + \varpi}{\tau}\right)^2 &= 1 + 2\varpi\tau^{-1} + (\varpi\tau^{-1})^2 \\ &= 2\varpi\tau^{-1}(1 + S\varpi\tau^{-1}) \quad \text{(Chapter III. Ex. 2)}, \end{aligned}$$

whence, to a scalar factor *près*, we have

$$\left(\frac{\varpi}{\tau}\right)^{\frac{1}{2}} = \frac{\tau + \varpi}{\tau}.$$

Hence, putting $\varpi = U(\beta + \alpha V\alpha\rho)$ and $\tau = U\rho$, we have from the first equation above

$$S\,.\,d\rho\,[U\rho + U(\beta + \alpha V\alpha\rho)] = 0.$$

But $$d(\beta + \alpha V\alpha\rho) = \alpha V\alpha d\rho = -d\rho - \alpha S\alpha d\rho,$$

and $$S.\alpha(\beta + \alpha V\alpha\rho) = 0,$$

so that we have finally

$$S.d\rho U\rho - S.d(\beta + \alpha V\alpha\rho)\, U(\beta + \alpha V\alpha\rho) = 0,$$

which is the differential of the second equation above. A curious particular case is a parabolic cylinder, as may be easily seen geometrically. The general surface has a parabolic section in the plane of $\alpha$, $\beta$; and a hyperbolic section in the plane of $\beta$, $\alpha\beta$.

It is easy to see that this is but a single case of a large class of integrable scalar functions, whose general type is

$$S.d\rho\left(\frac{\sigma - \rho}{\rho}\right)^{\frac{1}{2}}\rho = 0,$$

the equation of the reflecting surface; while

$$S(\sigma - \rho)\, d\sigma = 0$$

is the equation of the surface of the reflected wave: the integral of the former being, by the help of the latter, at once obtained in the form

$$T\rho \pm T(\sigma - \rho) = \text{constant}^*.$$

432. We next take Fresnel's *Theory of Double Refraction*, but merely for the purpose of shewing how quaternions simplify the processes required, and in no way to discuss the plausibility of the physical assumptions.

Let $t\varpi$ be the vector displacement of a portion of the ether, with the condition

$$\varpi^2 = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

the force of restitution, on Fresnel's assumption, is

$$t\,(a^2 iSi\varpi + b^2 jSj\varpi + c^2 kSk\varpi) = t\phi\varpi,$$

using the notation of Chapter V. Here the function $\phi$ is the *negative* of that of Chapter IX. (the force of restitution and the displacement being on the whole towards opposite parts), and it is clearly self-conjugate. $a^2$, $b^2$, $c^2$ are optical constants depending on the crystalline medium, and on the wave-length of the light, and may be considered as given.

Fresnel's second assumption is that the ether is incompressible, or that vibrations normal to a wave front are inadmissible. If, then, $\alpha$ be the unit normal to a plane wave in the crystal, we have of course

---

* *Proc. R. S. E.*, 1870–71.

$$\alpha^2 = -1 \qquad \text{.............................(2)},$$

and

$$S\alpha\varpi = 0 \qquad \text{.............................(3)};$$

but, and in addition, we have

$$\varpi^{-1}V\varpi\phi\varpi \parallel \alpha,$$

or

$$S.\alpha\varpi\phi\varpi = 0 \qquad \text{.............................(4)}.$$

This equation (4) is the embodiment of Fresnel's second assumption, but it may evidently be read as meaning, *the normal to the front, the direction of vibration, and that of the force of restitution are in one plane.*

433. Equations (3) and (4), if satisfied by $\varpi$, are also satisfied by $\varpi\alpha$, so that the plane (3) intersects the cone (4) in two lines at right angles to each other. That is, *for any given wave front there are two directions of vibration, and they are perpendicular to each other.*

434. The square of the normal speed of propagation of a plane wave is proportional to the ratio of the resolved part of the force of restitution in the direction of vibration, to the amount of displacement, hence

$$v^2 = S\varpi\phi\varpi.$$

Hence Fresnel's *Wave-surface* is the envelop of the plane

$$S\alpha\rho = \sqrt{S\varpi\phi\varpi} \qquad \text{.......................(5)},$$

with the conditions

$$\varpi^2 = -1 \qquad \text{.............................(1)},$$

$$\alpha^2 = -1 \qquad \text{.............................(2)},$$

$$S\alpha\varpi = 0 \qquad \text{.............................(3)},$$

$$S.\alpha\varpi\phi\varpi = 0 \qquad \text{.............................(4)}.$$

Formidable as this problem appears, it is easy enough. From (3) and (4) we get at once,

$$x\varpi = V.\alpha V\alpha\phi\varpi.$$

Hence, operating by $S.\varpi$,

$$-x = -S\varpi\phi\varpi = -v^2.$$

Therefore

$$(\phi + v^2)\varpi = -\alpha S\alpha\phi\varpi,$$

and

$$S.\alpha(\phi + v^2)^{-1}\alpha = 0 \qquad \text{....................(6)}.$$

In passing, we may remark that *this equation gives the normal speeds of the two rays whose fronts are perpendicular to* $\alpha$. In Cartesian cöordinates it is the well-known equation

$$\frac{l^2}{a^2-v^2}+\frac{m^2}{b^2-v^2}+\frac{n^2}{c^2-v^2}=0.$$

By this elimination of $\varpi$, our equations are reduced to

$$S.\alpha(\phi+v^2)^{-1}\alpha=0 \quad\ldots\ldots(6),$$

$$v=-S\alpha\rho \quad\ldots\ldots(5),$$

$$\alpha^2=-1 \quad\ldots\ldots(2).$$

They give at once, by § 326,

$$(\phi+v^2)^{-1}\alpha+v\rho S\alpha(\phi+v^2)^{-2}\alpha=h\alpha.$$

Operating by $S.\alpha$ we have

$$v^2S\alpha(\phi+v^2)^{-2}\alpha=h.$$

Substituting for $h$, and remarking that

$$S\alpha(\phi+v^2)^{-2}\alpha=-T^2(\phi+v^2)^{-1}\alpha,$$

because $\phi$ is self-conjugate, we have

$$v(\phi+v^2)^{-1}\alpha=\frac{v\alpha-\rho}{\rho^2+v^2}.$$

This gives at once, by rearrangement,

$$v(\phi+v^2)^{-1}\alpha=(\phi-\rho^2)^{-1}\rho.$$

Hence
$$(\phi-\rho^2)^{-1}\rho=\frac{v\alpha-\rho}{\rho^2+v^2}.$$

Operating by $S.\rho$ on this equation we have

$$S\rho(\phi-\rho^2)^{-1}\rho=-1 \quad\ldots\ldots(7),$$

which is the required equation.

[It will be a good exercise for the student to translate the last ten formulae into Cartesian cöordinates. He will thus reproduce almost exactly the steps by which Archibald Smith* first arrived at a simple and symmetrical mode of effecting the elimination. Yet, as we shall presently see, the above process is far from being the shortest and easiest to which quaternions conduct us.]

435. The Cartesian form of the equation (7) is not the usual one. It is, of course,

$$\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}=-1.$$

---

* *Cambridge Phil. Trans.*, 1835.

But write (7) in the form

$$S.\rho\frac{\rho^2}{\phi-\rho^2}\rho=-\rho^2,$$

or

$$S.\rho\frac{\phi}{\phi-\rho^2}\rho=0,$$

and we have the usual expression

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0.$$

The last-written quaternion equation can also be put into either of the new forms

$$T\left(\frac{\phi}{\phi-\rho^2}\right)^{\frac{1}{2}}\rho=0,$$

or

$$T(\rho^{-2}-\phi^{-1})^{-\frac{1}{2}}\rho=0.$$

436. By applying the results of §§ 183, 184 we may introduce a multitude of new forms. We must confine ourselves to the most simple; but the student may easily investigate others by a process precisely similar to that which follows.

Writing the equation of the wave as

$$S\rho\,(\phi^{-1}+g)^{-1}\rho=0,$$

where we have

$$g=-\rho^{-2},$$

we see that it may be changed to

$$S\rho\,(\phi^{-1}+h)^{-1}\rho=0,$$

if

$$mS\rho\phi\rho=gh\rho^2=-h.$$

Thus the new form is

$$S\rho\,(\phi^{-1}-mS\rho\phi\rho)^{-1}\rho=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Here $m=\frac{1}{a^2b^2c^2},\quad S\rho\phi\rho=a^2x^2+b^2y^2+c^2z^2,$

and the equation of the wave in Cartesian coördinates is, putting

$$r_1^{\,2}=a^2x^2+b^2y^2+c^2z^2,$$

$$\frac{x^2}{b^2c^2-r_1^{\,2}}+\frac{y^2}{c^2a^2-r_1^{\,2}}+\frac{z^2}{a^2b^2-r_1^{\,2}}=0.$$

437. By means of equation (1) of last section we may easily prove Plücker's Theorem:—

*The Wave-Surface is its own reciprocal with respect to the ellipsoid whose equation is*

$$S\rho\phi^{\frac{1}{2}}\rho=\frac{1}{\sqrt{m}}.$$

The equation of the plane of contact of tangents to this surface from the point whose vector is $\rho$ is

$$S\varpi\phi^{\frac{1}{2}}\rho = \frac{1}{\sqrt{m}}.$$

The reciprocal of this plane, with respect to the unit-sphere about the origin, has therefore a vector $\sigma$ where

$$\sigma = \sqrt{m} \,.\, \phi^{\frac{1}{2}}\rho.$$

Hence $$\rho = \frac{1}{\sqrt{m}}\phi^{-\frac{1}{2}}\sigma,$$

and when this is substituted in the equation of the wave we have for the reciprocal (with respect to the unit-sphere) of the reciprocal of the wave with respect to the above ellipsoid,

$$S\,.\,\sigma\left(\phi - \frac{1}{m}S\sigma\phi^{-1}\sigma\right)^{-1}\sigma = 0.$$

This differs from the equation (1) of last section solely in having $\phi^{-1}$ instead of $\phi$, and (consistently with this) $1/m$ instead of $m$. Hence it represents the index-surface. The required reciprocal of the wave with reference to the ellipsoid is therefore the wave itself.

438. Hamilton has given a remarkably simple investigation of the form of the equation of the wave-surface, in his *Elements*, p. 736, which the reader may consult with advantage. The following is essentially the same, but several steps of the process, which a skilled analyst would not require to write down, are retained for the benefit of the learner.

Let $$S\mu\rho = -1 \quad \text{..........................(1)}$$

be the equation of any tangent plane to the wave, i.e. of any wave-front. Then $\mu$ is the vector of wave-slowness, and the normal velocity of propagation is therefore $1/T\mu$. Hence, if $\varpi$ be the vector direction of displacement, $\mu^{-2}\varpi$ is the effective component of the force of restitution. Hence, $\phi\varpi$ denoting the whole force of restitution, we have

$$\phi\varpi - \mu^{-2}\varpi \parallel \mu,$$

or $$\varpi \parallel (\phi - \mu^{-2})^{-1}\mu,$$

and, as $\varpi$ is in the plane of the wave-front,

$$S\mu\varpi = 0,$$

or $$S\mu(\phi - \mu^{-2})^{-1}\mu = 0 \quad \text{.....................(2).}$$

This is, in reality, equation (6) of § 434. It appears here, however, as the equation of the *Index-Surface*, the polar reciprocal of the wave with respect to a unit-sphere about the origin. Of course the optical part of the problem is now solved, all that remains being the geometrical process of § 328.

439. Equation (2) of last section may be at once transformed, by the process of § 435, into

$$S\mu\,(\mu^2 - \phi^{-1})^{-1}\,\mu = 1.$$

Let us employ an auxiliary vector

$$\tau = (\mu^2 - \phi^{-1})^{-1}\mu,$$

whence $$\mu = (\mu^2 - \phi^{-1})\tau \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation now becomes

$$S\mu\tau = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

or, by (1), $$\mu^2\tau^2 - S\tau\phi^{-1}\tau = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Differentiating (3), subtract its half from the result obtained by operating with $S.\tau$ on the differential of (1). The remainder is

$$\tau^2\,S\mu d\mu - S\tau d\mu = 0.$$

But we have also (§ 328)

$$S\rho d\mu = 0,$$

and therefore, since $d\mu$ has an infinite number of values,

$$x\rho = \mu\tau^2 - \tau,$$

where $x$ is a scalar.

This equation, with (2), shews that

$$S\tau\rho = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Hence, operating on it by $S.\rho$, we have by (1) of last section

$$x\rho^2 = -\tau^2,$$

and therefore $$\rho^{-1} = -\mu + \tau^{-1}.$$

This gives $$\rho^{-2} = \mu^2 - \tau^{-2}.$$

Substituting from these equations in (1) above, it becomes

$$\tau^{-1} - \rho^{-1} = (\rho^{-2} + \tau^{-2} - \phi^{-1})\,\tau,$$

or $$\tau = (\phi^{-1} - \rho^{-2})^{-1}\,\rho^{-1}.$$

Finally, we have for the required equation, by (4),

$$S\rho^{-1}\,(\phi^{-1} - \rho^{-2})^{-1}\,\rho^{-1} = 0,$$

or, by a transformation already employed,

$$S\rho\,(\phi - \rho^2)^{-1}\rho = -1.$$

440. It may assist the student in the *practice* of quaternion analysis, which is our main object, if we give a few of these investigations by a somewhat varied process.

Thus, in § 432, let us write as in § 180,

$$a^2 iSi\varpi + b^2 jSj\varpi + c^2 kSk\varpi = \lambda' S\mu'\varpi + \mu' S\lambda'\varpi - p'\varpi.$$

We have, by the same processes as in § 432,

$$S.\varpi\alpha\lambda' S\mu'\varpi + S.\varpi\alpha\mu' S\lambda'\varpi = 0.$$

This may be written, *so far as the generating lines we require are concerned,*

$$\left.\begin{aligned} &S.\varpi\alpha V.\lambda'\varpi\mu' = 0 = S.\varpi\alpha\lambda'\varpi\mu', \\ &\text{since } \varpi\alpha \text{ is a vector.} \\ &\text{Or we may write} \\ &S.\mu' V.\varpi\lambda'\varpi\alpha = 0 = S.\mu'\varpi\lambda'\varpi\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

Equations (1) denote two cones of the second order which pass through the intersections of (3) and (4) of § 432. Hence their intersections are the directions of vibration.

441. By (1) we have

$$S.\varpi\lambda'\varpi\alpha\mu' = 0.$$

Hence $\varpi\lambda'\varpi$, $\alpha$, $\mu'$ are coplanar; and, as $\varpi$ is perpendicular to $\alpha$, it is equally inclined to $V\lambda'\alpha$ and $V\mu'\alpha$.

For, if $L$, $M$, $A$ be the projections of $\lambda'$, $\mu'$, $\alpha$ on the unit sphere, $BC$ the great circle whose pole is $A$, we are to find for the projections of the values of $\varpi$ on the sphere points $P$ and $P'$, such that if $LP$ be produced till

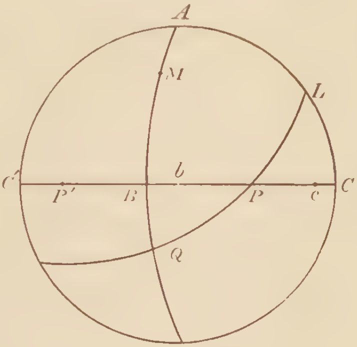
$$\widehat{PQ} = \widehat{LP},$$

$Q$ may lie on the great circle $AM$. Hence, evidently,

$$\widehat{CP} = \widehat{PB},$$

and $$\widehat{C'P'} = \widehat{P'B};$$

which proves the proposition, since the projections of $V\lambda'\alpha$ and $V\mu'\alpha$ on the sphere are points $b$ and $c$ in $BC$, distant by quadrants from $C$ and $B$ respectively.



442. Or thus, $S\varpi\alpha = 0$,

$$S\,.\,\varpi V\,.\,\alpha\lambda'\varpi\mu' = 0,$$

therefore $x\varpi = V\,.\,\alpha V\,.\,\alpha\lambda'\varpi\mu'$,

$$= -V\,.\,\lambda'\varpi\mu' - \alpha S\alpha V\,.\,\lambda'\varpi\mu'.$$

Hence $(S\lambda'\mu' - x)\varpi = (\lambda' + \alpha S\alpha\lambda')\,S\mu'\varpi + (\mu' + \alpha S\alpha\mu')\,S\lambda'\varpi.$

Operate by $S\,.\,\lambda'$, and we have

$$(x + S\lambda'\alpha S\mu'\alpha)\,S\lambda'\varpi = [\lambda'^2\alpha^2 - S^2\lambda'\alpha]\,S\mu'\varpi$$
$$= S\mu'\varpi T^2 V\lambda'\alpha.$$

Hence by symmetry,

$$\frac{S\mu'\varpi}{S\lambda'\varpi}T^2V\lambda'\alpha = \frac{S\lambda'\varpi}{S\mu'\varpi}T^2V\mu'\alpha,$$

or

$$\frac{S\lambda'\varpi}{TV\lambda'\alpha} \pm \frac{S\mu'\varpi}{TV\mu'\alpha} = 0,$$

$$S\varpi\left(\frac{\lambda'}{TV\lambda'\alpha} \pm \frac{\mu}{TV\mu'\alpha}\right) = 0,$$

and as $S\varpi\alpha = 0$,

$$\varpi = U\,(UV\lambda'\alpha \pm UV\mu'\alpha).$$

443. The optical interpretation of the common result of the last two sections is that *the planes of polarization of the two rays whose wave-fronts are parallel, bisect the angles contained by planes passing through the normal to the wave-front and the vectors* (optic axes) $\lambda'$, $\mu'$.

444. As in § 434, the normal speed is given by

$$v^2 = S\varpi\phi\varpi = 2S\lambda'\varpi S\mu'\varpi - p'\varpi^2$$
$$= p' \mp \frac{S^2\,.\,\lambda'\mu'\alpha}{(T \mp S)\,.\,V\lambda'\alpha V\mu'\alpha}.$$

[This transformation, effected by means of the value of $\varpi$ in § 442, is left to the reader.]

Hence, if $v_1$, $v_2$ be the velocities of the two waves whose normal is $\alpha$,

$$v_1^2 - v_2^2 = 2T\,.\,V\lambda'\alpha V\mu'\alpha$$
$$\propto \sin\widehat{\lambda'\alpha}\,\sin\widehat{\mu'\alpha}.$$

That is, *the difference of the squares of the speeds of the two waves varies as the product of the sines of the angles between the normal to the wave-front and the optic axes* ($\lambda'$, $\mu'$).

445. We have, obviously,

$$(T^2 - S^2) . V\lambda'\alpha V\mu'\alpha = T^2 V . V\lambda'\alpha V\mu'\alpha = S^2 . \lambda'\mu'\alpha.$$

Hence $$v^2 = p' \mp (T \pm S) . V\lambda'\alpha V\mu'\alpha.$$

The equation of the index surface, for which

$$T\rho = \frac{1}{v}, \qquad U\rho = \alpha,$$

is therefore $$1 = - p'\rho^2 \mp (T \pm S) . V\lambda'\rho V\mu'\rho.$$

This will, of course, become the equation of the reciprocal of the index-surface, i.e. the wave-surface, if we put for the function $\phi$ its reciprocal : i.e. if in the values of $\lambda'$, $\mu'$, $p'$ we put $1/a$, $1/b$, $1/c$ for $a$, $b$, $c$ respectively. We have then, and indeed it might have been deduced even more simply as a transformation of § 434 (7),

$$1 = - p\rho^2 \mp (T \pm S) . V\lambda\rho V\mu\rho,$$

as another form of the equation of Fresnel's wave.

If we employ the $\iota$, $\kappa$ transformation of § 128, this may be written, as the student may easily prove, in the form

$$(\kappa^2 - \iota^2)^2 = S^2 (\iota - \kappa)\rho + (TV\iota\rho \mp TV\kappa\rho)^2.$$

446. We may now, in furtherance of our object, which is to give varied examples of quaternions, not complete treatment of any one subject, proceed to deduce some of the properties of the wave-surface from the different forms of its equation which we have given.

447. *Fresnel's construction of the wave by points.*

From § 290 (4) we see at once that the lengths of the principal semidiameters of the central section of the ellipsoid

$$S\rho\phi^{-1}\rho = 1,$$

by the plane $$S\alpha\rho = 0,$$

are determined by the equation

$$S . \alpha (\phi^{-1} - \rho^{-2})^{-1} \alpha = 0.$$

If these lengths be laid off along $\alpha$, the central perpendicular to the cutting plane, their extremities lie on a surface for which $\alpha = U\rho$, and $T\rho$ has values determined by the equation.

Hence the equation of the locus is

$$S\rho (\phi^{-1} - \rho^{-2})^{-1} \rho = 0,$$

as in §§ 434, 439.

Of course the index-surface is derived from the reciprocal ellipsoid

$$S\rho\phi\rho = 1$$

by the same construction.

448. Again, in the equation

$$1 = -p\rho^2 \mp (T \pm S) . V\lambda\rho V\mu\rho,$$

suppose $V\lambda\rho = 0$, or $V\mu\rho = 0$,

we obviously have

$$\rho = \pm \frac{U\lambda}{\sqrt{p}} \quad \text{or} \quad \rho = \pm \frac{U\mu}{\sqrt{p}},$$

and there are therefore four singular points.

To find the nature of the surface near these points put

$$\rho = \frac{U\lambda}{\sqrt{p}} + \varpi,$$

where $T\varpi$ is very small, and reject terms above the first order in $T\varpi$. The equation of the wave becomes, in the neighbourhood of the singular point,

$$2pS\lambda\varpi + S . \varpi V . \lambda V\lambda\mu = \pm T . V\lambda\varpi V\lambda\mu,$$

which belongs to a cone of the second order.

449. From the similarity of its equation to that of the wave, it is obvious that the index-surface also has four conical cusps. As an infinite number of tangent planes can be drawn at such a point, the reciprocal surface must be capable of being touched by a plane at an infinite number of points; so that the wave-surface has four tangent planes which touch it along ridges.

To find their form, let us employ the last form of equation of the wave in § 445. If we put

$$TV\iota\rho = TV\kappa\rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have the equation of a cone of the second degree. It meets the wave at its intersections with the planes

$$S(\iota - \kappa)\rho = \pm(\kappa^2 - \iota^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now the wave-surface is *touched* by these planes, because we cannot have the quantity on the first side of this equation greater in absolute magnitude than that on the second, so long as $\rho$ satisfies the equation of the wave.

That the curves of contact are circles appears at once from (1) and (2), for they give in combination

$$\rho^2 = \mp S(\iota+\kappa)\rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the equations of two spheres on which the curves in question are situated.

The diameter of this circular ridge is

$$TV.(\iota+\kappa)U(\iota-\kappa) = \frac{2TV\iota\kappa}{T(\iota-\kappa)} = \frac{1}{b}\sqrt{(a^2-b^2)(b^2-c^2)}.$$

[Simple as these processes are, the student will find on trial that the equation

$$S\rho\,(\phi^{-1}-\rho^{-2})^{-1}\rho = 0,$$

gives the results quite as simply. For we have only to examine the cases in which $-\rho^{-2}$ has the value of one of the roots of the symbolical cubic in $\phi^{-1}$. In the present case $T\rho = b$ is the only one which requires to be studied.]

450. By § 438, we see that the auxiliary vector of the succeeding section, viz.

$$\tau = (\mu^2-\phi^{-1})^{-1}\mu = (\phi^{-1}-\rho^{-2})^{-1}\rho^{-1},$$

is parallel to the direction of the force of restitution, $\phi\varpi$. Hence, as Hamilton has shewn, the equation of the wave, in the form

$$S\tau\rho = 0,$$

(4) of § 439, indicates that *the direction of the force of restitution is perpendicular to the ray.*

Again, as for any one versor of a vector of the wave there are two values of the tensor, which are found from the equation

$$S.U\rho\,(\phi^{-1}-\rho^{-2})^{-1}U\rho = 0,$$

we see by § 447 that *the lines of vibration for a given plane front are parallel to the axes of any section of the ellipsoid*

$$S.\rho\phi^{-1}\rho = 1$$

*made by a plane parallel to the front; or to the tangents to the lines of curvature at a point where the tangent plane is parallel to the wave-front.*

451. Again, *a curve which is drawn on the wave-surface so as to touch at each point the corresponding line of vibration* has

$$\phi d\rho \parallel (\phi^{-1}-\rho^{-2})^{-1}\rho.$$

Hence $S\phi\rho d\rho = 0$, or $S\rho\phi\rho = C$,

so that such curves are the intersections of the wave with a series of ellipsoids concentric with it.

452. For *curves cutting at right angles the lines of vibration* we have

$$d\rho \parallel V\rho\phi^{-1}(\phi^{-1} - \rho^{-2})^{-1}\rho$$

$$\parallel V\rho(\phi - \rho^2)^{-1}\rho.$$

Hence $S\rho d\rho = 0$, or $T\rho = C$,

so that the curves in question lie on concentric spheres.

They are also *spherical conics*, because where

$$T\rho = C$$

the equation of the wave becomes

$$S.\rho(\phi^{-1} + C^{-2})^{-1}\rho = 0,$$

the equation of a cyclic cone, whose vertex is at the common centre of the sphere and the wave-surface, and which cuts them in their curve of intersection. (§§ 432—452, *Quarterly Math. Journal*, 1859.) The student may profitably compare, with the preceding investigations, the (generally) very different processes which Hamilton (in his *Elements*) applies to this problem.

### *E. Electrodynamics.*

453. As another example we take the case of the action of electric currents on one another or on magnets; and the mutual action of permanent magnets.

A comparison between the processes we employ and those of Ampère (*Théorie des Phénomènes Electrodynamiques*) will at once shew how much is gained in simplicity and directness by the use of quaternions.

The same gain in simplicity will be noticed in the investigations of the mutual effects of permanent magnets, where the resultant forces and couples are at once introduced in their most natural and direct forms.

454. Ampère's experimental laws may be stated as follows:

I. Equal and opposite currents in the same conductor produce equal and opposite effects on other conductors.

II. The effect of a conductor bent or twisted in any manner is equivalent to that of a straight one, provided that the two are traversed by equal currents, and the former *nearly* coincides with the latter.

III. No closed circuit can set in motion an element of a circular conductor about an axis through the centre of the circle and perpendicular to its plane.

IV. In similar systems traversed by equal currents the forces are equal.

To these we add the assumption that the action between two elements of currents is in the straight line joining them. [In a later section (§ 473) other assumptions will be made in place of this.] We also take for granted that the effect of any element of a current on another is directly as the product of the strengths of the currents, and of the lengths of the elements.

455. Let there be two closed currents whose strengths are $a$ and $a_1$; let $\alpha'$, $\alpha_1$ be elements of these, $\alpha$ being the vector joining their middle points. Then the effect of $\alpha'$ on $\alpha_1$ must, when resolved along $\alpha_1$, be a complete differential with respect to $\alpha$ (i.e. with respect to the three independent variables involved in $\alpha$), since the total resolved effect of the closed circuit of which $\alpha'$ is an element is zero by III.

Also by I, II, the effect is a function of $T\alpha$, $S\alpha\alpha'$, $S\alpha\alpha_1$, and $S\alpha'\alpha_1$, since these are sufficient to resolve $\alpha'$ and $\alpha_1$ into elements parallel and perpendicular to each other and to $\alpha$. Hence the mutual effect is

$$aa_1 U\alpha f(T\alpha,\ S\alpha\alpha',\ S\alpha\alpha_1,\ S\alpha'\alpha_1),$$

and the resolved effect parallel to $\alpha_1$ is

$$aa_1 SU\alpha_1 U\alpha f.$$

Also, that action and reaction may be equal in absolute magnitude, $f$ must be symmetrical in $S\alpha\alpha'$ and $S\alpha\alpha_1$. Again, $\alpha'$ (as differential of $\alpha$) *can* enter *only to the first power*, and *must* appear in each term of $f$.

Hence $$f = AS\alpha'\alpha_1 + BS\alpha\alpha' S\alpha\alpha_1.$$

But, by IV, this must be independent of the dimensions of the system. Hence $A$ is of $-2$ and $B$ of $-4$ dimensions in $T\alpha$. Therefore

$$\frac{1}{T\alpha}\{AS\alpha\alpha_1 S\alpha'\alpha_1 + BS\alpha\alpha' S^2\alpha\alpha_1\}$$

is a complete differential, with respect to $\alpha$, if $d\alpha = \alpha'$. Let

$$A = \frac{C}{T\alpha^2},$$

where $C$ is a constant depending on the units employed, therefore

$$d\frac{C}{2T\alpha^3} = \frac{B}{T\alpha} S\alpha\alpha',$$

or
$$B = \tfrac{3}{2}\frac{C}{T\alpha^4},$$

and the resolved effect

$$= \frac{Caa_1}{2Ta_1} d \frac{S^2\alpha\alpha_1}{T\alpha^3} = Caa_1 \frac{S\alpha\alpha_1}{T\alpha_1 T\alpha^5} (-\alpha^2 S\alpha'\alpha_1 + \tfrac{3}{2} S\alpha\alpha' S\alpha\alpha_1)$$

$$= Caa_1 \frac{S\alpha\alpha_1}{T\alpha_1 T\alpha^5} (S.V\alpha\alpha' V\alpha\alpha_1 + \tfrac{1}{2} S\alpha\alpha' S\alpha\alpha_1).$$

The factor in brackets is evidently proportional in the ordinary notation to

$$\sin\theta \sin\theta' \cos\omega - \tfrac{1}{2}\cos\theta\cos\theta'.$$

456. Thus the whole force is

$$\frac{Caa_1\alpha}{2S\alpha\alpha_1} d \frac{S^2\alpha\alpha_1}{T\alpha^3} = \frac{Caa_1\alpha}{2S\alpha\alpha'} d_1 \frac{S^2\alpha\alpha'}{T\alpha^3},$$

as we should expect, $d_1\alpha$ being $= \alpha_1$. [This may easily be transformed into

$$-\frac{2Caa_1 U\alpha}{(T\alpha)^{\frac{1}{2}}} dd_1 (T\alpha)^{\frac{1}{2}},$$

which is the quaternion expression for Ampère's well-known form.]

457. The whole effect on $\alpha_1$ of the closed circuit, of which $\alpha'$ is an element, is therefore

$$\frac{Caa_1}{2}\int \frac{\alpha}{S\alpha\alpha_1} d \frac{(S\alpha\alpha_1)^2}{T\alpha^3},$$

$$= \frac{Caa_1}{2}\left\{\frac{\alpha S\alpha\alpha_1}{T\alpha^3} - V.\alpha_1 \int \frac{V\alpha\alpha'}{T\alpha^3}\right\}$$

between proper limits. As the integrated part is the same at both limits, the effect is

$$-\frac{Caa_1}{2} V\alpha_1\beta, \text{ where } \beta = \int \frac{V\alpha\alpha'}{T\alpha^3} = \int \frac{dU\alpha}{\alpha},$$

and depends on the form of the closed circuit.

458. This vector $\beta$, which is of great importance in the whole theory of the effects of closed or indefinitely extended circuits, corresponds to the line which is called by Ampère "*directrice de l'action électrodynamique.*" It has a definite value at each point of space, independent of the existence of any other current.

Consider the circuit a polygon whose sides are indefinitely small; join its angular points with any assumed point, erect at the latter, perpendicular to the plane of each elementary triangle so formed, a vector whose length is $\omega/r$, where $\omega$ is the vertical angle of the triangle and $r$ the length of one of the containing sides; the sum of such vectors is the "*directrice*" at the assumed point.

[We may anticipate here so far as to give another expression for this important vector in terms of processes to be explained later.

We have, by the formula (for a closed curve) of § 497 below,

$$\beta = \int V\frac{\rho d\rho}{T\rho^3} = \int V d\rho \nabla \frac{1}{T\rho} = \iint ds\, V.V(U\nu\nabla)\nabla\frac{1}{T\rho},$$

(where $ds$ is an element of any surface bounded by the circuit, $U\nu$ its unit normal)

$$= \iint ds\, U\nu\nabla^2 \frac{1}{T\rho} - \nabla \iint ds S U\nu \nabla \frac{1}{T\rho}.$$

But the last integral is obviously the whole spherical angle, $\Omega$ suppose, subtended by the circuit at the origin, and (unless $T\rho = 0$) we have (§ 145)

$$\nabla^2 \frac{1}{T\rho} = 0.$$

Hence, generally,

$$\beta = -\nabla\Omega.$$

Thus $\Omega$ may be considered as representing a potential, for which $\beta$ is the corresponding force.

This is a "many-valued" function, altering by $4\pi$ whenever we pass through a surface closing the circuit. For if $\sigma$ be the vector of a closed curve, the work done against $\beta$ during the circuit is

$$\int S\beta d\sigma = -\int S d\sigma \nabla\Omega = \int d\Omega.$$

The last term is zero if the curve is not linked with the circuit, but increases by $\pm 4\pi$ for each linkage with the circuit.]

459. The mere *form* of the result of § 457 shews at once that *if the element* $\alpha_1$ *be turned about its middle point, the direction of the resultant action is confined to the plane whose normal is* $\beta$.

Suppose that the element $\alpha_1$ is forced to remain perpendicular to some given vector $\delta$, we have

$$S\alpha_1\delta = 0,$$

and the whole action in its plane of motion is proportional to

$$TV.\delta V\alpha_1\beta.$$

But $$V.\delta V\alpha_1\beta = -\alpha_1 S\beta\delta.$$

Hence the action is evidently constant for all possible positions of $\alpha_1$; or

*The effect of any system of closed currents on an element of a conductor which is restricted to a given plane is (in that plane) independent of the direction of the element.*

460. Let the closed current be *plane* and *very small.* Let $\epsilon$ (where $T\epsilon = 1$) be its normal, and let $\gamma$ be the vector of any point within it (as the centre of inertia of its area); the middle point of $\alpha_1$ being the origin of vectors.

Let $$\alpha = \gamma + \rho;\text{ therefore } \alpha' = \rho',$$

and $$\beta = \int \frac{V\alpha\alpha'}{T\alpha^3} = \int \frac{V(\gamma+\rho)\rho'}{T(\gamma+\rho)^3}$$

$$= \frac{1}{T\gamma^3}\int V(\gamma+\rho)\rho'\left\{1 + \frac{3S\gamma\rho}{T\gamma^2}\right\}$$

to a sufficient approximation.

Now (between limits) $\int V\rho\rho' = 2A\epsilon$,

where $A$ is the area of the closed circuit.

Also generally

$$\int V\gamma\rho' S\gamma\rho = \tfrac{1}{2}(S\gamma\rho V\gamma\rho + \gamma V.\gamma\int V\rho\rho')$$

$$= \text{(between limits) } A\gamma V\gamma\epsilon.$$

Hence for this case

$$\beta = \frac{A}{T\gamma^3}\left(2\epsilon + \frac{3\gamma V\gamma\epsilon}{T\gamma^2}\right)$$

$$= -\frac{A}{T\gamma^3}\left(\epsilon + \frac{3\gamma S\gamma\epsilon}{T\gamma^2}\right).$$

461. If, instead of one small plane closed current, there be a series of such, of equal area, disposed regularly in a tubular form, let $x$ be the distance between two consecutive currents measured along the axis of the tube; then, putting $\gamma' = x\epsilon$, we have for the whole effect of such a set of currents on $\alpha_1$

$$\frac{CAaa_1}{2x} V . \alpha_1 \int \left( \frac{\gamma'}{T\gamma^3} + \frac{3\gamma S\gamma\gamma'}{T\gamma^5} \right)$$

$$= \frac{CAaa_1}{2x} \frac{V\alpha_1\gamma}{T\gamma^3} \text{ (between proper limits).}$$

If the axis of the tubular arrangement be a closed curve this will evidently vanish. Hence *a closed solenoid exerts no influence on an element of a conductor. The same is evidently true if the solenoid be indefinite in both directions.*

If the axis extend to infinity in one direction, and $\gamma_0$ be the vector of the other extremity, the effect is

$$\frac{CAaa_1}{2x} \frac{V\alpha_1\gamma_0}{T\gamma_0^3},$$

and is therefore *perpendicular to the element and to the line joining it with the extremity of the solenoid. It is evidently inversely as* $T\gamma_0^2$ *and directly as the sine of the angle contained between the direction of the element and that of the line joining the latter with the extremity of the solenoid. It is also inversely as* x, *and therefore directly as the number of currents in a unit of the axis of the solenoid.*

462. To find the effect of the whole circuit whose element is $\alpha_1$ on the extremity of the solenoid, we must change the sign of the above and put $\alpha_1 = \gamma_0'$; therefore the effect is

$$-\frac{CAaa_1}{2x} \int \frac{V\gamma_0'\gamma_0}{T\gamma_0^3},$$

an integral of the species considered in § 458, whose value is easily assigned in particular cases.

463. *Suppose the conductor to be straight, and indefinitely extended in both directions.*

Let $h\theta$ be the vector perpendicular to it from the extremity of the solenoid, and let the conductor be $\parallel \eta$, where $T\theta = T\eta = 1$.

Therefore $\gamma_0 = h\theta + y\eta$ (where $y$ is a scalar),

$$V\gamma_0'\gamma_0 = hy' V\eta\theta,$$

and the integral in § 462 is

$$hV\eta\theta\int_{-\infty}^{+\infty}\frac{y'}{(h^2+y^2)^{\frac{3}{2}}}=\frac{2}{h}V\eta\theta.$$

The whole effect is therefore

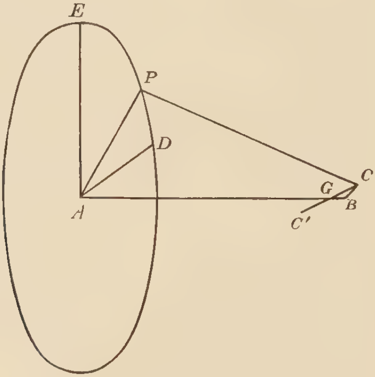
$$-\frac{CAaa_1}{xh}V\eta\theta,$$

and is thus *perpendicular to the plane passing through the conductor and the extremity of the solenoid, and varies inversely as the distance of the latter from the conductor.*

This is exactly the observed effect of an indefinite straight current on a magnetic pole, or particle of free magnetism.

464. *Suppose the conductor to be circular, and the pole nearly in its axis.* [This is *not* a proper subject for Quaternions.]

Let $EPD$ be the conductor, $AB$ its axis, and $C$ the pole; $BC$ perpendicular to $AB$, and small in comparison with $AE=h$ the radius of the circle.



Let $AB$ be $a_1 i$, $BC=bk$, $AP=h(jx+ky)$

where $\begin{matrix}x\\y\end{matrix}\Big\}=\Big\{\begin{matrix}\cos\\\sin\end{matrix}\Big\}\angle EAP=\Big\{\begin{matrix}\cos\\\sin\end{matrix}\Big\}\theta.$

Then $CP=\gamma=a_1 i+bk-h(jx+ky).$

And the effect on $C$ $\propto\int\frac{V\gamma\gamma'}{T\gamma^3},$

$$\propto h\int\frac{\theta'\{(h-by)i+a_1xj+a_1yk\}}{(a_1^{\,2}+b^2+h^2-2bhy)^{\frac{3}{2}}},$$

where the integral extends to the whole circuit.

465. Suppose in particular $C$ to be one pole of a small magnet or solenoid $CC'$ whose length is $2l$, and whose middle point is at $G$ and distant $a$ from the centre of the conductor.

Let $\angle CGB = \Delta$. Then evidently

$$a_1 = a + l \cos \Delta,$$
$$b = l \sin \Delta.$$

Also the effect on $C$ becomes, if $a_1{}^2 + b^2 + h^2 = A^2$,

$$\frac{h}{A^3}\int \theta' \left\{(h - by)\, i + a_1 x j + a_1 y k\right\} \left\{1 + \frac{3hby}{A^2} + \frac{15}{2}\frac{h^2b^2y^2}{A^4} + \ldots\right\}$$
$$= \frac{\pi h^2}{A^3}\left(2i - \frac{3b^2 i}{A^2} + \frac{3a_1 bk}{A^2} + \frac{15}{2}\frac{h^2b^2 i}{A^4} + \ldots\right),$$

since for the whole circuit

$$\int \theta' y^{2n} = 2\pi \frac{(2n)!}{2^{2n}(n!)^2},$$
$$\int \theta' y^{2n+1} = 0,$$
$$\int \theta' x y^m = 0.$$

If we suppose the centre of the magnet fixed, the vector axis of the couple produced by the action of the current on $C$ is

$$l V . (i \cos \Delta + k \sin \Delta) \int \frac{V \gamma \gamma'}{T \gamma^3}$$
$$\propto \frac{\pi h^2 l \sin \Delta}{A^3} j \left\{2 - \frac{3b^2}{A^2} + \frac{15}{2}\frac{h^2 b^2}{A^4} - \frac{3a_1 b \cos \Delta}{A^2 \sin \Delta}\right\}.$$

If $A$, &c. be now developed in powers of $l$, this at once becomes

$$\frac{\pi h^2 l \sin \Delta}{(a^2 + h^2)^{\frac{3}{2}}} j \left\{2 - \frac{6al \cos \Delta}{a^2 + h^2} + \frac{15 a^2 l^2 \cos^2 \Delta}{(a^2 + h^2)^2} - \frac{3l^2}{a^2 + h^2}\right.$$
$$\left. - \frac{3l^2 \sin^2 \Delta}{a^2 + h^2} + \frac{15}{2}\frac{h^2 l^2 \sin^2 \Delta}{(a^2 + h^2)^2} - 3\frac{(a + l \cos \Delta) l \cos \Delta}{a^2 + h^2}\left(1 - \frac{5al \cos \Delta}{a^2 + h^2}\right)\right\}.$$

Putting $-l$ for $l$ and changing the sign of the whole to get that for pole $C'$, we have for the vector axis of the complete couple

$$\frac{4\pi h^2 l \sin \Delta}{(a^2 + h^2)^{\frac{3}{2}}} j \left\{1 + \frac{3}{4}\frac{l^2(4a^2 - h^2)(4 - 5\sin^2 \Delta)}{(a^2 + h^2)^2} + \text{\&c.}\right\},$$

which is almost exactly proportional to $\sin \Delta$, if $2a$ be nearly equal to $h$ and $l$ be small. (See Ex. 15 at end of Chapter.)

On this depends a modification of the tangent galvanometer. (Bravais, *Ann. de Chimie*, xxxviii. 309.)

466. As before, the effect of an indefinite solenoid on $\alpha_1$ is

$$\frac{CAaa_1}{2x}\frac{V\alpha_1\gamma}{T\gamma^3}.$$

Now suppose $\alpha_1$ to be an element of a small plane circuit, $\delta$ the vector of the centre of inertia of its area, the pole of the solenoid being origin.

Let $\gamma = \delta + \rho$, then $\alpha_1 = \rho'$.

The whole effect is therefore

$$-\frac{CAaa_1}{2x}\int\frac{V(\delta+\rho)\rho'}{T(\delta+\rho)^3}$$

$$=\frac{CAA_1aa_1}{2xT\delta^3}\left(\epsilon_1+\frac{3\delta S\delta\epsilon_1}{T\delta^2}\right),$$

where $A_1$ and $\epsilon_1$ are, for the new circuit, what $A$ and $\epsilon$ were for the former.

Let the new circuit also belong to an indefinite solenoid, and let $\delta_0$ be the vector joining the poles of the two solenoids. Then the mutual effect is

$$\frac{CAA_1aa_1}{2xx_1}\int\left(\frac{\delta'}{T\delta^3}+\frac{3\delta S\delta\delta'}{T\delta^5}\right)$$

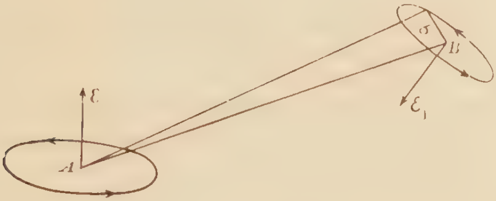
$$=\frac{CAA_1aa_1}{2xx_1}\frac{\delta_0}{(T\delta_0)^3}\propto\frac{U\delta_0}{(T\delta_0)^2},$$

which is exactly *the mutual effect of two magnetic poles. Two finite solenoids, therefore, act on each other exactly as two magnets, and the pole of an indefinite solenoid acts as a particle of free magnetism.*

467. The mutual attraction of two indefinitely small plane closed circuits, whose normals are $\epsilon$ and $\epsilon_1$, may evidently be deduced by twice differentiating the expression $U\delta/T\delta^2$ for the mutual action of the poles of two indefinite solenoids, making $d\delta$ in one differentiation $\parallel \epsilon$ and in the other $\parallel \epsilon_1$.

But it may also be calculated directly by a process which will give us in addition the couple impressed on one of the circuits by the other, supposing for simplicity the first to be *circular.* [In the sketch we are supposed to be looking at the faces turned *towards* one another.]

Let $A$ and $B$ be the centres of inertia of the areas of $A$ and $B$,



$\epsilon$ and $\epsilon_1$ vectors normal to their planes, $\sigma$ any vector radius of $B$, $AB = \beta$.

Then whole effect on $\sigma'$, by §§ 457, 460,

$$\propto \frac{A}{T(\beta+\sigma)^3} V\sigma' \left\{ \epsilon + \frac{3(\beta+\sigma)S(\beta+\sigma)\epsilon}{T(\beta+\sigma)^2} \right\},$$

$$\propto \frac{1}{T\beta^3} \left\{ V\sigma'\epsilon \left(1 + \frac{3S\beta\sigma}{T\beta^2}\right) + \frac{3V\sigma'\beta S\beta\epsilon}{T\beta^2} \left(1 + \frac{5S\beta\sigma}{T\beta^2}\right) \right.$$
$$\left. + \frac{3V\sigma'\beta S\sigma\epsilon}{T\beta^2} + 3\frac{V\sigma'\sigma S\beta\epsilon}{T\beta^2} \right\}.$$

But between proper limits,

$$\int V\sigma'\eta S\theta\sigma = -A_1 V.\eta V\theta\epsilon_1,$$

for generally $\int V\sigma'\eta S\theta\sigma = -\frac{1}{2}(V\eta\sigma S\theta\sigma + V.\eta V.\theta \int V\sigma\sigma')$.

Hence, after a reduction or two, we find that the whole force exerted by $A$ on the centre of inertia of the area of $B$

$$\propto \frac{AA_1}{T\beta^5} \left\{ \beta \left( S\epsilon\epsilon_1 + \frac{5S\beta\epsilon S\beta\epsilon_1}{T\beta^2} \right) + \epsilon S\beta\epsilon_1 + \epsilon_1 S\beta\epsilon \right\}.$$

This, as already observed, may be at once found by twice differentiating $\dfrac{U\beta}{T\beta^2}$. In the same way the vector moment, due to $A$, about the centre of inertia of $B$,

$$\propto \frac{A}{T\beta^3} \int V.\sigma \left( V\sigma'\epsilon + \frac{3V\sigma'\beta S\beta\epsilon}{T\beta^2} \right),$$

$$\propto -\frac{AA_1}{T\beta^3} \left( V\epsilon\epsilon_1 + \frac{3V\beta\epsilon_1 S\beta\epsilon}{T\beta^2} \right).$$

These expressions for the whole force of one small magnet on the centre of inertia of another, and the couple about the latter, seem to be the simplest that can be given. It is easy to deduce from them the ordinary forms. For instance, the whole resultant couple on the second magnet

$$\propto \frac{T\left(V\epsilon\epsilon_1 + \frac{3V\beta\epsilon_1 S\beta\epsilon}{T\beta^2}\right)}{T\beta^3},$$

may easily be shewn to coincide with that given by Ellis (*Camb. Math. Journal*, iv. 95), though it seems to lose in simplicity and capability of interpretation by such modifications.

468. The above formulae shew that the whole force exerted by one small magnet *M*, on the centre of inertia of another *m*, consists of four terms which are, in order,

1st. *In the line joining the magnets, and proportional to the cosine of their mutual inclination.*

2nd. *In the same line, and proportional to five times the product of the cosines of their respective inclinations to this line.*

3rd and 4th. *Parallel to* $\begin{Bmatrix}\mathrm{m}\\ \mathrm{M}\end{Bmatrix}$ *and proportional to the cosine of the inclination of* $\begin{Bmatrix}\mathrm{M}\\ \mathrm{m}\end{Bmatrix}$ *to the joining line.*

All these forces are, in addition, inversely as the fourth power of the distance between the magnets.

For the couples about the centre of inertia of *m* we have

1st. *A couple whose axis is perpendicular to each magnet, and which is as the sine of their mutual inclination.*

2nd. *A couple whose axis is perpendicular to* m *and to the line joining the magnets, and whose moment is as three times the product of the sine of the inclination of* m, *and the cosine of the inclination of* M, *to the joining line.*

In addition these couples vary inversely as the third power of the distance between the magnets.

[These results afford a good example of what has been called the *internal* nature of the methods of quaternions, reducing, as they do at once, the forces and couples to others independent of any lines of reference, other than those necessarily belonging to the system under consideration. To shew their ready applicability, let us take a Theorem due to Gauss.]

469. *If two small magnets be at right angles to each other, the moment of rotation of the first is approximately twice as great when the axis of the second passes through the centre of the first, as when the axis of the first passes through the centre of the second.*

In the first case $\epsilon \parallel \beta \perp \epsilon_1$;

therefore moment $= \frac{C'}{T\beta^3} T(\epsilon\epsilon_1 - 3\epsilon\epsilon_1) = \frac{2C'}{T\beta^3} T\epsilon\epsilon_1$.

In the second $\epsilon_1 \parallel \beta \perp \epsilon$;

therefore moment $= \frac{C'}{T\beta^3} T\epsilon\epsilon_1$. Hence the theorem.

470. Again, we may easily reproduce the results of § 467, if for the two small circuits we suppose two small magnets perpendicular to their planes to be substituted. $\beta$ is then the vector joining the middle points of these magnets, and by changing the tensors we may take $2\epsilon$ and $2\epsilon_1$ as the vector lengths of the magnets.

Hence evidently the mutual effect

$$\propto \frac{U}{T^2}(\beta + \epsilon - \epsilon_1) - \frac{U}{T^2}(\beta - \epsilon - \epsilon_1) + \frac{U}{T^2}(\beta - \epsilon + \epsilon_1) - \frac{U}{T^2}(\beta + \epsilon + \epsilon_1),$$

which is easily reducible to

$$-\frac{12}{T\beta^5}\left\{\beta\left(S\epsilon\epsilon_1 + \frac{5S\beta\epsilon S\beta\epsilon_1}{T\beta^2}\right) + \epsilon_1 S\beta\epsilon + \epsilon S\beta\epsilon_1\right\},$$

as before, if smaller terms be omitted.

If we operate with $V.\,\epsilon_1$ on the two first terms of the unreduced expression, and take the difference between this result and the same with the sign of $\epsilon_1$ changed, we have the whole vector axis of the couple on the magnet $2\epsilon_1$, which is therefore, as before, seen to be proportional to

$$\frac{4}{T\beta^3}\left(V\epsilon_1\epsilon + \frac{3V\epsilon_1\beta S\beta\epsilon}{T\beta^2}\right).$$

471. Let $F(\gamma)$ be the potential of any system upon a unit particle at the extremity of $\gamma$. Then we have

$$S\nu d\gamma = 0,$$

where $\nu$ is a vector normal giving the force in direction and magnitude (§ 148).

Now by § 460 we have for the vector force exerted by a small plane closed circuit on a particle of free magnetism the expression

$$-\frac{A}{T\gamma^3}\left(\epsilon+\frac{3\gamma S\gamma\epsilon}{T\gamma^2}\right),$$

merging in $A$ the factors depending on the strength of the current and the strength of magnetism of the particle.

Hence the potential is

$$\propto A\int\frac{1}{T\gamma^3}\left(S\epsilon d\gamma+\frac{3S\gamma d\gamma S\gamma\epsilon}{T\gamma^2}\right),$$

$$\propto\frac{AS\epsilon\gamma}{T\gamma^3},$$

$$\propto\frac{\text{area of circuit projected perpendicular to }\gamma}{T\gamma^2},$$

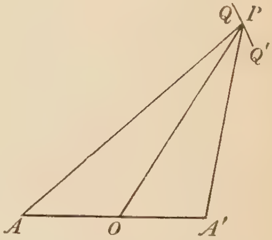
$$\propto\text{spherical opening subtended by circuit.}$$

The constant is omitted in the integration, as the potential must evidently vanish for infinite values of $T\gamma$.

By means of Ampère's idea of breaking up a finite circuit into an indefinite number of indefinitely small ones, it is evident that the above result may be at once extended to the case of such a finite closed circuit.

472. Quaternions give a simple method of deducing the well-known property of the *Magnetic Curves.*

Let $A$, $A'$ be two equal magnetic poles, whose vector distance, $2\alpha$, is bisected in $O$, $QQ'$ an indefinitely small magnet whose length is $2\rho'$, where $\rho=OP$. Then evidently, taking moments,



$$\frac{V(\rho+\alpha)\rho'}{T(\rho+\alpha)^3}=\pm\frac{V(\rho-\alpha)\rho'}{T(\rho-\alpha)^3},$$

where the upper or lower sign is to be taken according as the poles are like or unlike.

[This equation may also be obtained at once by differentiating the equation of the equipotential surface,

$$\frac{1}{T(\rho+\alpha)}\mp\frac{1}{T(\rho-\alpha)}=\text{const.},$$

and taking $\rho'$ parallel to its normal (§ 148).]

Operate by $S.V\alpha\rho$,

$$\frac{S\alpha\rho'(\rho+\alpha)^2 - S\alpha(\rho+\alpha)S\rho'(\rho+\alpha)}{T(\rho+\alpha)^3} = \pm\,\{\text{same with } -\alpha\},$$

or $$S.\alpha V\left(\frac{\rho'}{\rho+\alpha}\right)U(\rho+\alpha) = \pm\,\{\text{same with } -\alpha\},$$

i.e. $$S\alpha dU(\rho+\alpha) = \pm\, S\alpha dU(\rho-\alpha),$$

$$S\alpha\{U(\rho+\alpha) \mp U(\rho-\alpha)\} = \text{const.},$$

or $$\cos\angle OAP \pm \cos\angle OA'P = \text{const.},$$

the property referred to.

If the poles be unequal, one of the terms to the left must be multiplied by the ratio of their strengths.

(§§ 453—472, *Quarterly Math. Journal*, 1860.)

## *F. General Expressions for the Action between Linear Elements.*

473. The following general investigation of different possible expressions for the mutual action between elements of linear conductors is taken from *Proc. R. S. E.* 1873—4.

Ampère's data for closed currents are briefly given in § 454 above, and are here referred to as I, II, III, IV, respectively.

(*a*) First, let us investigate the expression for the *force* exerted by one element on another.

Let $\alpha$ be the vector joining the elements $\alpha_1$, $\alpha'$, of two circuits; then, by I, II, the action of $\alpha_1$ on $\alpha'$ is *linear* in each of $\alpha_1$, $\alpha'$, and may, therefore, be expressed as

$$\phi\alpha',$$

where $\phi$ is a linear and vector function, into each of whose constituents $\alpha_1$ enters linearly.

The resolved part of this along $\alpha'$ is

$$S.U\alpha'\phi\alpha',$$

and, by III, this must be a complete differential as regards the circuit of which $\alpha_1$ is an element. Hence,

$$\phi\alpha' = -(S.\alpha_1\nabla)\,\psi\alpha' + V\alpha'\chi\alpha_1,$$

where $\psi$ and $\chi$ are linear and vector functions whose constituents involve $\alpha$ only. That this is the case follows from the fact that $\phi\alpha'$ is homogeneous and linear in each of $\alpha_1$, $\alpha'$. It farther follows, from IV, that the part of $\phi\alpha'$ which does not disappear after integration round each of the closed circuits is of no dimensions in $T\alpha$, $T\alpha'$, $T\alpha_1$. Hence $\chi$ is of $-2$ dimensions in $T\alpha$, and thus

$$\chi\alpha_1 = \frac{p\alpha S\alpha\alpha_1}{T\alpha^4} + \frac{q\alpha_1}{T\alpha^2} + \frac{rV\alpha\alpha_1}{T\alpha^3},$$

where $p$, $q$, $r$ are numbers.

Hence we have

$$\phi\alpha' = -S(\alpha_1\nabla)\psi\alpha' + \frac{pV\alpha'\alpha S\alpha\alpha_1}{T\alpha^4} + \frac{qV\alpha'\alpha_1}{T\alpha^2} + \frac{rV.\alpha'V\alpha\alpha_1}{T\alpha^3}.$$

Change the sign of $\alpha$ in this, and interchange $\alpha'$ and $\alpha_1$, and we get the action of $\alpha'$ on $\alpha_1$. This, with $\alpha'$ and $\alpha_1$ again interchanged, and the sign of the whole changed, should reproduce the original expression—since the effect depends on the relative, not the absolute, positions of $\alpha$, $\alpha_1$, $\alpha'$. This gives at once,

$$p = 0, \quad q = 0,$$

and

$$\phi\alpha' = -S(\alpha_1\nabla)\psi\alpha' + \frac{rV.\alpha'V\alpha\alpha_1}{T\alpha^3},$$

with the condition that the first term changes its sign with $\alpha$, and thus that

$$\psi\alpha' = \alpha S\alpha\alpha'\mathsf{F}(T\alpha) + \alpha' F(T\alpha),$$

which, by change of $\mathsf{F}$, may be written

$$= \alpha S(\alpha'\nabla) f(T\alpha) + \alpha' F(T\alpha),$$

where $f$ and $F$ are any scalar functions whatever.

Hence

$$\phi\alpha' = -S(\alpha_1\nabla)[\alpha S(\alpha'\nabla) f(T\alpha) + \alpha' F(T\alpha)] + \frac{rV.\alpha'V\alpha\alpha_1}{T\alpha^3},$$

which is the general expression required.

(*b*) The simplest possible form for the action of one current-element on another is, therefore,

$$\phi\alpha' = \frac{rV.\alpha'V\alpha\alpha_1}{T\alpha^3}.$$

Here it is to be observed that Ampère's *directrice* for the circuit $\alpha_1$ is

$$\theta = \int \frac{V\alpha\alpha_1}{T\alpha^3},$$

the integral extending round the circuit; so that, finally,

$$\phi\alpha' = -rS\alpha_1\nabla . V\alpha'\theta.$$

(*c*) We may obtain from the general expression above the absolutely symmetrical form,

$$\frac{rV.\alpha'\alpha\alpha_1}{T\alpha^3},$$

if we assume

$$f(T\alpha) = \text{const.}, \quad F(T\alpha) = \frac{r}{T\alpha}.$$

Here the action of $\alpha'$ on $\alpha_1$ is parallel and equal to that of $\alpha_1$ on $\alpha'$. The forces, in fact, form a couple, for $\alpha$ is to be taken negatively for the second—and their common direction is the vector drawn to the corner $a$ of a spherical triangle $abc$, whose sides $ab$, $bc$, $ca$ in order are bisected by the extremities of the vectors $U\alpha'$, $U\alpha$, $U\alpha_1$. Compare Hamilton's *Lectures on Quaternions*, §§ 223—227.

(*d*) To obtain Ampère's form for the effect of one element on another write, in the general formula above,

$$f(T\alpha) = \frac{r}{T\alpha}, \quad F(T\alpha) = 0,$$

and we have

$$\frac{1}{r}\phi\alpha' = -S\alpha_1\nabla.\left[-\frac{\alpha S\alpha\alpha'}{T\alpha^3}\right] + \frac{V.\alpha' V\alpha\alpha_1}{T\alpha^3},$$

$$= -\frac{\alpha_1 S\alpha\alpha'}{T\alpha^3} - \frac{\alpha S\alpha_1\alpha'}{T\alpha^3} - \frac{3\alpha S\alpha\alpha' S\alpha\alpha_1}{T\alpha^5} + \frac{V.\alpha' V\alpha\alpha_1}{T\alpha^3},$$

$$= +\frac{2\alpha}{T\alpha^5}\left(\alpha^2 S\alpha_1\alpha' - \frac{3}{2} S\alpha\alpha' S\alpha\alpha_1\right),$$

$$= -\frac{2\alpha}{T\alpha^5}\left(S.V\alpha\alpha' V\alpha\alpha_1 + \frac{1}{2} S\alpha\alpha' S\alpha\alpha_1\right),$$

which are the usual forms.

(*e*) The remainder of the expression, containing the arbitrary terms, is of course still of the form

$$-S(\alpha_1\nabla)[\alpha S(\alpha'\nabla) f(T\alpha) + \alpha' F(T\alpha)].$$

In the ordinary notation this expresses a force whose components are proportional to

(1) Along $\alpha$ $\qquad -r\dfrac{d^2 f}{ds_1 ds'}$.

(Note that, in *this* expression, $r$ is the distance between the elements.)

(2) Parallel to $\alpha'$ $\qquad \dfrac{dF}{ds_1}$.

(3) Parallel to $\alpha_1$ $\quad -\dfrac{df}{ds'}$.

If we assume $f = F = -Q$, we obtain the result given by Clerk-Maxwell (*Electricity and Magnetism*, § 525), which differs from the above only because he assumes that the force exerted by one element on another, when the first is parallel and the second perpendicular to the line joining them, is *equal* to that exerted when the first is perpendicular and the second parallel to that line.

($f$) What precedes is, of course, only a particular case of the following interesting problem :—

*Required the most general expression for the mutual action of two rectilinear elements, each of which has dipolar symmetry in the direction of its length, and which may be resolved and compounded according to the usual kinematical law.*

The data involved in this statement are equivalent to I and II of Ampère's data above quoted. Hence, keeping the same notation as in ($a$) above, the force exerted by $\alpha_1$ on $\alpha'$ must be expressible as

$$\phi\alpha'$$

where $\phi$ is a linear and vector function, whose constituents are linear and homogeneous in $\alpha_1$; and, besides, involve only $\alpha$.

By interchanging $\alpha_1$ and $\alpha'$, and changing the sign of $\alpha$, we get the force exerted by $\alpha'$ on $\alpha_1$. If in this we again interchange $\alpha_1$ and $\alpha'$, and change the sign of the whole, we must obviously reproduce $\phi\alpha'$. Hence we must have $\phi\alpha'$ changing its sign with $\alpha$, or

$$\phi\alpha' = P\alpha S\alpha_1\alpha' + Q\alpha S\alpha\alpha_1 S\alpha\alpha' + R\alpha_1 S\alpha\alpha' + \mathsf{R}\alpha' S\alpha\alpha_1,$$

where $P$, $Q$, $R$, $\mathsf{R}$ are functions of $T\alpha$ only.

($g$) The vector *couple* exerted by $\alpha_1$ on $\alpha'$ must obviously be expressible in the form

$$V.\alpha'\varpi\alpha_1,$$

where $\varpi$ is a new linear and vector function depending on $\alpha$ alone. Hence its most general form is

$$\varpi\alpha_1 = P\alpha_1 + Q\alpha S\alpha\alpha_1,$$

where $P$ and $Q$ are functions of $T\alpha$ only. The form of these functions, whether in the expression for the force or for the couple, depends on the special data for each particular case. Symmetry shews that there is no term such as

$$RV\alpha\alpha_1.$$

(*h*) As an example, let $\alpha_1$ and $\alpha'$ be elements of solenoids or of uniformly and linearly magnetised wires, it is obvious that, as a closed solenoid or ring-magnet exerts no external action,

$$\phi\alpha' = -S\alpha_1\nabla \,.\, \psi\alpha'.$$

Thus we have introduced a different datum in place of Ampère's No. III. But in the case of solenoids the Third Law of Newton holds—hence

$$\phi\alpha' = S\alpha_1\nabla S\alpha'\nabla \,.\, \chi\alpha,$$

where $\chi$ is a linear and vector function, and can therefore be of no other form than

$$\alpha f(T\alpha).$$

Now two solenoids, each extended to infinity in one direction, act on one another like two magnetic poles, so that (this being our equivalent for Ampère's datum No. IV.)

$$\chi\alpha = p\frac{\alpha}{T\alpha^3}.$$

Hence the vector force exerted by one small magnet on another is

$$pS\alpha_1\nabla S\alpha'\nabla\frac{\alpha}{T\alpha^3}.$$

(*i*) For the couple exerted by one element of a solenoid, or of a uniformly and longitudinally magnetised wire, on another, we have of course the expression

$$V \,.\, \alpha'\varpi\alpha_1,$$

where $\varpi$ is some linear and vector function.

Here, in the first place, it is obvious that

$$\varpi\alpha_1 = -S\alpha_1\nabla \,.\, \frac{\alpha}{F(T\alpha)};$$

for the couple vanishes for a closed circuit of which $\alpha_1$ is an element, and the integral of $\varpi\alpha_1$ must be a linear and vector function of $\alpha$ alone. It is easy to see that in this case

$$F(T\alpha) \propto (T\alpha)^3.$$

(*j*) If, again, $\alpha_1$ be an element of a solenoid, and $\alpha'$ an element of current, the force is

$$\phi\alpha' = -S\alpha_1\nabla \,.\, \psi\alpha',$$

where

$$\psi\alpha' = P\alpha' + Q\alpha S\alpha\alpha' + RV\alpha\alpha'.$$

But no portion of a solenoid can produce a force on an element of current in the direction of the element, so that

$$\phi\alpha' = V.\alpha'\chi\alpha_1,$$

whence $$P=0, \quad Q=0,$$

and we have $$\phi\alpha' = -S\alpha_1\nabla\,(RV\alpha\alpha').$$

This must be of $-1$ linear dimensions when we integrate for the effect of one pole of a solenoid, so that

$$R = \frac{p}{T\alpha^3}.$$

If the current be straight and infinite each way, its equation being

$$\alpha = \beta + x\gamma,$$

where $$T\gamma = 1 \text{ and } S\beta\gamma = 0,$$

we have, for the whole force exerted on it by the pole of a solenoid, the expression

$$p\beta\gamma\int_{-\infty}^{+\infty}\frac{dx}{(T\beta^2+x^2)^{\frac{3}{2}}} = -2p\beta^{-1}\gamma,$$

which agrees with known facts.

(*k*) Similarly, for the couple produced by an element of a solenoid on an element of a current we have

$$V\alpha'\varpi\alpha_1,$$

where $$\varpi\alpha_1 = -S\alpha_1\nabla.\psi\alpha,$$

and it is easily seen that

$$\psi\alpha = \frac{r\alpha}{T\alpha^3}.$$

(*l*) In the case first treated, the couple exerted by one current-element on another is, by (*g*),

$$V.\alpha'\varpi\alpha_1,$$

where, of course, $\pm\varpi\alpha_1$ are the vector forces applied at either end of $\alpha'$. Hence the work done when $\alpha'$ changes its direction is

$$-S.\delta\alpha'\varpi\alpha_1,$$

with the condition $$S.\alpha'\delta\alpha' = 0.$$

So far, therefore, as change of direction of $\alpha'$ alone is concerned, the mutual potential energy of the two elements is of the form

$$S.\alpha'\varpi\alpha_1.$$

This gives, by the expression for $\varpi$ in ($g$), the following value

$$PS\alpha'\alpha_1 + QS\alpha\alpha'S\alpha\alpha_1.$$

Hence, integrating round the circuit of which $\alpha_1$ is an element, we have (§ 495 below)

$$\int (PS\alpha'\alpha_1 + QS\alpha\alpha'S\alpha\alpha_1) = \iint ds_1 S\,.\,U\nu_1\nabla(P\alpha' + Q\alpha S\alpha\alpha'),$$

$$= \iint ds_1 S\,.\,U\nu_1\left(\frac{\alpha\alpha'P'}{T\alpha} - \alpha'\alpha Q\right),$$

$$= \iint ds_1 S\,.\,U\nu_1 V\alpha\alpha'\Phi,$$

where $$\Phi = \frac{P'}{T\alpha} + Q.$$

Integrating this round the other circuit we have for the mutual potential energy of the two, so far as it depends on the expression above, the value

$$\iint ds_1 S\,.\,U\nu_1 \int V\alpha\alpha'\Phi$$

$$= -\iint ds_1 S\,.\,U\nu_1 \iint ds' V\,.\,V(U\nu'\nabla)\,\alpha\Phi$$

$$= \iint ds_1 \iint ds' \left\{S\,.\,U\nu_1 U\nu'\,(2\Phi + T\alpha\Phi') + S\alpha U\nu' S\alpha U\nu_1 \frac{\Phi'}{T\alpha}\right\}.$$

But, by Ampère's result, that two closed circuits act on one another as two magnetic shells, it should be

$$\iint ds_1 \iint ds' S\,.\,U\nu_1\nabla S\,.\,U\nu'\nabla \frac{1}{T\alpha}$$

$$= \iint ds_1 \iint ds' \left(S\,.\,U\nu_1 U\nu' \frac{1}{T\alpha^3} + 3S\alpha U\nu' S\alpha U\nu_1 \frac{1}{T\alpha^5}\right).$$

Comparing, we have

$$\left.\begin{aligned} \frac{1}{T\alpha^3} &= 2\Phi + T\alpha\Phi' \\ \frac{3}{T\alpha^3} &= T\alpha\Phi' \end{aligned}\right\},$$

giving $$\Phi = -\frac{1}{T\alpha^3}, \quad \Phi' = \frac{3}{T\alpha^4},$$

which are consistent with one another, and which lead to

$$\frac{P'}{T\alpha} + Q = -\frac{1}{T\alpha^3}.$$

Hence, if we put

$$Q = \frac{1-n}{2nT\alpha^3},$$

we get $$P = \frac{1+n}{2nT\alpha},$$

and the mutual potential of two elements is of the form

$$(1+n)\frac{S\alpha'\alpha_1}{T\alpha}+(1-n)\frac{S\alpha\alpha' S\alpha\alpha_1}{T\alpha^3},$$

which is the expression employed by Helmholtz in his paper *Ueber die Bewegungsgleichungen der Electricität, Crelle,* 1870, p. 76.

### *G. Application of* $\nabla$ *to certain Physical Analogies.*

474. The chief elementary results into which $\nabla$ enters, in connection with displacements, are given in § 384 above. The following are direct applications.

Thus, if $\sigma$ be the vector-displacement of that point of a homogeneous elastic solid whose vector is $\rho$, we have, $p$ being the consequent pressure produced,

$$\nabla p+\nabla^2\sigma=0 \text{ ........................ } (1),$$

whence $S\delta\rho\nabla^2\sigma=-S\delta\rho\nabla p=\delta p$, a complete differential ...... (2).

Also, generally, $p=cS\nabla\sigma$,

and if the solid be incompressible

$$S\nabla\sigma=0 \text{ ............................ } (3).$$

Thomson has shewn (*Camb. and Dub. Math. Journal,* ii. p. 62), that the forces produced by given distributions of matter, electricity, magnetism, or galvanic currents, can be represented at every point by displacements of such a solid producible by external forces. It may be useful to give his analysis, with some additions, in a quaternion form, to shew the insight gained by the simplicity of the present method.

475. Thus, if $S\sigma\delta\rho=\delta\,\frac{1}{T\rho}$, we may write each equal to

$$-S\delta\rho\nabla\frac{1}{T\rho}.$$

This gives $\sigma=-\nabla\frac{1}{T\rho}$,

the vector-force exerted by one particle of matter or free electricity on another. This value of $\sigma$ evidently satisfies (2) and (3).

Again, if $S.\delta\rho\nabla\sigma=\delta\frac{S\alpha\rho}{T\rho^3}$, either is equal to

$$-S.\delta\rho\nabla\frac{V\alpha\rho}{T\rho^3} \text{ by (4) of § 146.}$$

Here a particular case is

$$\sigma = -\frac{V\alpha\rho}{T\rho^3},$$

which is the vector-force exerted by an element $\alpha$ of a current upon a particle of magnetism at $\rho$. (§ 461.)

476. Also, by § 146 (3),

$$\nabla\frac{V\alpha\rho}{T\rho^3} = \frac{\alpha\rho^2 - 3\rho S\alpha\rho}{T\rho^5},$$

and we see by §§ 460, 461 that this is the vector-force exerted by a small plane current at the origin (its plane being perpendicular to $\alpha$) upon a magnetic particle, or pole of a solenoid, at $\rho$. This expression, being a pure vector, denotes an elementary rotation caused by the distortion of the solid, and it is evident that the above value of $\sigma$ satisfies the equations (2), (3), and the distortion is therefore producible by external forces. Thus the effect of an element of a current on a magnetic particle is expressed directly by the displacement, while that of a small closed current or magnet is represented by the vector-axis of the rotation caused by the displacement.

477. Again, let $S\delta\rho\nabla^2\sigma = \delta\frac{S\alpha\rho}{T\rho^3}$.

It is evident that $\sigma$ satisfies (2), and that the right-hand side of the above equation may be written

$$-S\,.\,\delta\rho\nabla\frac{V\alpha\rho}{T\rho^3}.$$

Hence a particular case is

$$\nabla\sigma = -\frac{V\alpha\rho}{T\rho^3},$$

and this satisfies (3) also.

Hence the corresponding displacement is producible by external forces, and $\nabla\sigma$ is the rotation axis of the element at $\rho$, and is seen as before to represent the vector-force exerted on a particle of magnetism at $\rho$ by an element $\alpha$ of a current at the origin.

478. It is interesting to observe that a particular value of $\sigma$ in this case is

$$\sigma = -\tfrac{1}{2}\nabla S\alpha U\rho - \frac{\alpha}{T\rho},$$

as may easily be proved by substitution.

Again, if $$S\delta\rho\sigma = -\delta\frac{S\alpha\rho}{T\rho^3},$$

we have evidently $$\sigma = \nabla\frac{S\alpha\rho}{T\rho^3}.$$

Now, as $\frac{S\alpha\rho}{T\rho^3}$ is the potential of a small magnet $\alpha$, at the origin, on a particle of free magnetism at $\rho$, $\sigma$ is the resultant magnetic force, and represents also a possible distortion of the elastic solid by external forces, since $\nabla\sigma = \nabla^2\sigma = 0$, and thus (2) and (3) are both satisfied. (*Proc. R. S. E.* 1862.)

### *H. Elementary Properties of* $\nabla$.

479. In the next succeeding sections we commence with a form of definition of the operator $\nabla$ somewhat different from that of Hamilton (§ 145), as we shall thus entirely avoid the use of Cartesian cöordinates. For this purpose we write

$$S.\alpha\nabla = -d_\alpha,$$

where $\alpha$ is any unit-vector, the meaning of the right-hand operator (neglecting its sign) being the *rate of change of the function to which it is applied* per unit of length in the direction of the vector $\alpha$. If $\alpha$ be not a unit-vector we may treat it as a vector-velocity, and then the right-hand operator means the *rate of change per unit of time* due to the change of position.

Let $\alpha$, $\beta$, $\gamma$ be any rectangular system of unit-vectors, then by a fundamental quaternion transformation

$$\nabla = -\alpha S\alpha\nabla - \beta S\beta\nabla - \gamma S\gamma\nabla = \alpha d_\alpha + \beta d_\beta + \gamma d_\gamma,$$

which is identical with Hamilton's form so often given above. (*Lectures*, § 620.)

480. This mode of viewing the subject enables us to see at once that $\nabla$ is an *Invariant*, and that the effect of applying it to any scalar function of the position of a point is to give its *vector of most rapid increase.* Hence, when it is applied to a potential $u$, we have the corresponding vector-force. From a velocity-potential we obtain the velocity of the fluid element at $\rho$; and from the temperature of a conducting solid we obtain the temperature-

gradient in the direction of the flux of heat. Finally, whatever series of surfaces is represented by

$$u = C,$$

the vector $\nabla u$ is the normal at the point $\rho$, and its length is inversely as the normal distance at that point between two consecutive surfaces of the series.

Hence it is evident that

$$S \,.\, d\rho \nabla u = - du,$$

or, as it may be written,

$$- S \,.\, d\rho \nabla = d\,;$$

the left-hand member therefore expresses total differentiation in virtue of any arbitrary, but small, displacement $d\rho$.

These results have been already given above, but they were not obtained in such a direct manner.

Many very curious and useful transformations may easily be derived (see Ex. 34, Chap. XI.) from the assumption

$$d\sigma = \phi d\rho, \text{ or } \phi\alpha = - S\alpha\nabla \,.\, \sigma,$$

where the constituents of $\phi$ are known functions of $\rho$. For instance, if we write

$$\nabla_\sigma = i\frac{d}{d\xi} + j\frac{d}{d\eta} + k\frac{d}{d\zeta},$$

where $\sigma = i\xi + j\eta + k\zeta,$

we find at once $\nabla = \phi' \nabla_\sigma, \text{ or } \nabla_\sigma = \phi'^{-1} \nabla\,;$

a formula which contains the whole basis of the theory of the change of independent variables from $x, y, z$ to $\xi, \eta, \zeta$, or *vice versâ*. The reader may easily develop this application. Its primary interest is, of course, purely mathematical:—but it has most important uses in applied mathematics. Our limits, however, do not permit us to reach the regions of its special physical usefulness.

481. To interpret the operator $V . \alpha\nabla$, let us apply it to a potential function $u$. Then we easily see that $u$ may be taken under the vector sign, and the expression

$$V(\alpha\nabla)\, u = V.\, \alpha\nabla u$$

denotes the vector-couple due to the force at $\rho$ about a point whose relative vector is $\alpha$.

Again, if $\sigma$ be any vector function of $\rho$, we have by ordinary quaternion operations

$$V(\alpha\nabla)\,.\,\sigma = S\,.\,\alpha V\nabla\sigma + \alpha S\nabla\sigma - \nabla S\alpha\sigma.$$

The meaning of the third term (in which it is of course understood that $\nabla$ operates on $\sigma$ alone) is obvious from what precedes. The other terms were explained in § 384.

## *J. Applications of* $\nabla$ *to Line-, Surface-, and Volume-Integrals.*

482. In what follows we have constantly to deal with integrals extended over a closed surface, compared with others taken through the space enclosed by such a surface; or with integrals over a limited surface, compared with others taken round its bounding curve. The notation employed is as follows. If $Q$ per unit of length, of surface, or of volume, at the point $\rho$, $Q$ being any quaternion, be the quantity to be summed, these sums will be denoted by

$$\iint Q ds \text{ and } \iiint Q d\varsigma,$$

when comparing integrals over a closed surface with others through the enclosed space; and by

$$\iint Q ds \text{ and } \int Q T d\rho,$$

when comparing integrals over an unclosed surface with others round its boundary. No ambiguity is likely to arise from the double use of

$$\iint Q ds,$$

for its meaning in any case will be obvious from the integral with which it is compared. What follows is mainly from *Trans. R. S. E.* 1869—70. See also *Proc. R. S. E.* 1862—3.

483. We have shewn in § 384 that, if $\sigma$ be the vector displacement of a point originally situated at

$$\rho = ix + jy + kz,$$

then

$$S\,.\,\nabla\sigma$$

expresses the increase of density of aggregation of the points of the system caused by the displacement.

484. Suppose, now, space to be uniformly filled with points, and a closed surface $\Sigma$ to be drawn, through which the points can freely move when displaced.

Then it is clear that the increase of number of points within the space $\Sigma$, caused by a displacement, may be obtained by either of two processes—by taking account of the increase of density at all points within $\Sigma$, or by estimating the excess of those which pass inwards through the surface over those which pass outwards. These are the principles usually employed (for a mere element of volume) in forming the so-called 'Equation of Continuity.'

Let $\nu$ be the normal to $\Sigma$ at the point $\rho$, drawn outwards, then we have at once (by equating the two different expressions of the same quantity above explained) the equation

$$\iiint S . \nabla \sigma d\varsigma = \iint S . \sigma U\nu ds,$$

which is our fundamental equation so long as we deal with triple integrals. [It will be shewn later (§ 500) that the corresponding relation between the single and the double integral can be deduced directly from this.]

As a first and very simple example of its use, let $\rho$ be written for $\sigma$. It becomes

$$-3 \iiint d\varsigma = \iint S\rho U\nu ds,$$

i.e. the volume of any closed space is the sum of the elements of area of its surface, each multiplied by one-third of the perpendicular from the origin on its plane.

485. Next, suppose $\sigma$ to represent the vector force exerted upon a unit particle at $\rho$ (of ordinary matter, electricity, or magnetism) by any distribution of attracting matter, electricity, or magnetism partly outside, partly inside $\Sigma$. Then, if $P$ be the potential at $\rho$,

$$\sigma = \nabla P,$$

and if $r$ be the density of the attracting matter, &c., at $\rho$,

$$\nabla \sigma = \nabla^2 P = 4\pi r$$

by Poisson's extension of Laplace's equation.

Substituting in the fundamental equation, we have

$$4\pi \iiint r d\varsigma = 4\pi M = \iint S . \nabla P U\nu ds,$$

where $M$ denotes the whole quantity of matter, &c., inside $\Sigma$. This is a well-known theorem.

486. Let $P$ and $P_1$ be any scalar functions of $\rho$, we can of course find the distribution of matter, &c., requisite to make either

of them the potential at $\rho$; for, if the necessary densities be $r$ and $r_1$ respectively, we have as before

$$\nabla^2 P = 4\pi r, \quad \nabla^2 P_1 = 4\pi r_1.$$

Now
$$\nabla (P\nabla P_1) = \nabla P \nabla P_1 + P\nabla^2 P_1.$$

Hence, if in the formula of § 484 we put

$$\sigma = P\nabla P_1,$$

we obtain

$$\iiint S . \nabla P \nabla P_1 d\varsigma = -\iiint P\nabla^2 P_1 d\varsigma + \iint PS . \nabla P_1 U\nu ds,$$
$$= -\iiint P_1 \nabla^2 P d\varsigma + \iint P_1 S . \nabla P U\nu ds,$$

which are the common forms of Green's Theorem. Sir W. Thomson's extension of it follows at once from the same proof.

487. If $P_1$ be a many-valued function, but $\nabla P_1$ single-valued, and if $\Sigma$ be a multiply-connected* space, the above expressions require a modification which was first shewn to be necessary by Helmholtz, and first supplied by Thomson. For simplicity, suppose $\Sigma$ to be doubly-connected (as a ring or endless rod, whether knotted or not). Then if it be cut through by a surface $s$, it will become simply-connected, but the surface-integrals have to be increased by terms depending upon the portions just added to the whole surface. In the first form of Green's Theorem, just given, the only term altered is the last: and it is obvious that if $p_1$ be the increase of $P_1$ after a complete circuit of the ring, the portion to be added to the right-hand side of the equation is

$$p_1 \iint S . \nabla P U\nu ds,$$

taken over the cutting surface only. A similar modification is easily seen to be produced by each additional complexity in the space $\Sigma$.

488. The immediate consequences of Green's theorem are well known, so that I take only a few examples.

Let $P$ and $P_1$ be the potentials of one and the same distribution of matter, and let none of it be within $\Sigma$. Then we have

$$\iiint (\nabla P)^2 d\varsigma = \iint PS . \nabla P U\nu ds,$$

---

* Called by Helmholtz, after Riemann, *mehrfach zusammenhängend*. In translating Helmholtz's paper (*Phil. Mag.* 1867) I used the above as an English equivalent. Sir W. Thomson in his great paper on *Vortex Motion* (*Trans. R. S. E.* 1868) uses the expression "multiply-continuous."

so that if $\nabla P$ is zero all over the surface of $\Sigma$, it is zero all through the interior, i.e., the potential is constant inside $\Sigma$. If $P$ be the velocity-potential in the irrotational motion of an incompressible fluid, this equation shews that there can be no such motion of the fluid unless there is a normal motion at some part of the bounding surface, so long at least as $\Sigma$ is simply-connected.

Again, if $\Sigma$ is an equipotential surface,

$$\iiint(\nabla P)^2 d\varsigma = P\iint S.\nabla PU\nu ds = P\iiint\nabla^2 Pd\varsigma$$

by the fundamental theorem. But there is by hypothesis no matter inside $\Sigma$, so this shews that the potential is constant throughout the interior. Thus there can be no equipotential surface, not including some of the attracting matter, within which the potential can change. Thus it cannot have a maximum or minimum value at points unoccupied by matter.

489. Again, in an isotropic body whose thermal conductivity does not vary with temperature, the equation of heat-conduction is

$$\frac{dv}{dt} + \frac{\mathbf{k}}{c}\nabla^2 v = 0,$$

where (for the moment) $\mathbf{k}$ and $c$ represent as usual the conductivity, and the water equivalent of unit volume.

The surface condition (assuming Newton's Law of Cooling) is

$$\mathbf{k}S.U\nu\nabla v - hv = 0.$$

Assuming, after Fourier, that a particular integral is

$$v = \epsilon^{-mt}u,$$

we have $$\nabla^2 u - \frac{mc}{\mathbf{k}}u = 0, \qquad S.U\nu\nabla u - \frac{h}{\mathbf{k}}u = 0.$$

Let $u_m$ be a particular integral of the first of these linear differential equations. Substitute it for $u$ in the second; and we obtain (with the aid of the equation of the bounding surface) a scalar equation giving the admissible values of $m$.

Suppose the distribution of temperature when $t = 0$ to be given; it may be expressed linearly in terms of the various values of $u_m$, thus

$$w = u_{t=0} = \Sigma A_m u_m.$$

For if $u_1$, $u_2$ be any two of these particular integrals, we have by Green's Theorem, and the differential equations,

$$\frac{m_1 c}{\mathbf{k}}\iiint u_1 u_2 d\varsigma - \frac{h}{\mathbf{k}}\iint u_1 u_2 ds = \frac{m_2 c}{\mathbf{k}}\iiint u_2 u_1 d\varsigma - \frac{h}{\mathbf{k}}\iint u_2 u_1 ds.$$

Hence, unless $m_1 = m_2$, we have

$$\iiint u_1 u_2 d\varsigma = 0.$$

Thus we have $$A_m \iiint u_m^2 d\varsigma = \iiint w u_m d\varsigma.$$

$A_m$ being thus found, we have generally

$$v = \Sigma A_m \epsilon^{-mt} u_m.$$

490. If, in the fundamental theorem, we suppose

$$\sigma = \nabla \tau,$$

which imposes the condition that

$$S\nabla\sigma = 0,$$

i.e., that the $\sigma$ displacement is effected without condensation, it becomes

$$\iint S\,.\,\nabla\tau U\nu ds = \iiint S\nabla^2 \tau d\varsigma = 0.$$

Suppose any closed curve to be traced on the surface $\Sigma$, dividing it into two parts. This equation shews that the surface-integral is the same for both parts, the difference of sign being due to the fact that the normal is drawn in opposite directions on the two parts. Hence we see that, with the above limitation of the value of $\sigma$, the double integral is the same for all surfaces bounded by a given closed curve. It must therefore be expressible by a single integral taken round the curve. The value of this integral will presently be determined (§ 495).

491. The theorem of § 485 may be written

$$\iiint \nabla^2 P d\varsigma = \iint S U\nu \nabla P ds = \iint S\,(U\nu\nabla)\,P ds.$$

From this we conclude at once that if

$$\sigma = \alpha P + \beta P_1 + \gamma P_2,$$

(which may, of course, represent any vector whatever) we have

$$\iiint \nabla^2 \sigma d\varsigma = \iint S\,(U\nu\nabla)\,\sigma ds,$$

or, if $$\nabla^2 \sigma = \tau,$$

$$\iiint \tau d\varsigma = \iint S\,(U\nu\nabla^{-1})\,\tau ds.$$

This gives us the means of representing, by a surface-integral, a vector-integral taken through a definite space. We have already seen how to do the same for a scalar-integral—so that we can now express in this way, subject, however, to an ambiguity presently to be mentioned, the general integral

$$\iiint q d\varsigma,$$

where $q$ is any quaternion whatever. It is evident that it is only in certain classes of cases that we can *expect* a perfectly definite expression of such a volume-integral in terms of a surface-integral.

492. In the above formula for a vector-integral there may present itself an ambiguity introduced by the inverse operation

$$\nabla^{-1}$$

to which we must devote a few words. The assumption

$$\nabla^2\sigma = \tau$$

is tantamount to saying that, if the constituents of $\sigma$ are the potentials of certain distributions of matter, &c., those of $\tau$ are the corresponding densities each multiplied by $4\pi$.

If, therefore, $\tau$ be given throughout the space enclosed by $\Sigma$, $\sigma$ is given by this equation *so far only* as it depends upon the distribution within $\Sigma$, and must be completed by an arbitrary vector depending on *three* potentials of mutually independent distributions exterior to $\Sigma$.

But, if $\sigma$ be given, $\tau$ is perfectly definite; and as

$$\nabla\sigma = \nabla^{-1}\tau,$$

the value of $\nabla^{-1}$ is also completely defined. These remarks must be carefully attended to in using the theorem above: since they involve as particular cases of their application many curious theorems in Fluid Motion, &c.

493. As a very special case, the equation

$$V\nabla\sigma = 0$$

of course gives $\nabla\sigma = u$, a scalar.

Now, if $v$ be the potential of a distribution whose density is $u$, we have

$$\nabla^2 v = 4\pi u.$$

We know that when $u$ is assigned this equation gives one, and but one, definite value for $v$. We have in fact, by the definition of a potential,

$$v = \iiint \frac{u_1 ds}{T(\rho - \rho_1)},$$

where the integration (confined to $u_1$ and $\rho_1$) extends to all space in which $u$ differs from zero.

Thus there is no ambiguity in

$$v = 4\pi\nabla^{-2}u,$$

and therefore $$\sigma = \frac{1}{4\pi}\nabla v$$

is also determinate.

494. This shews the nature of the arbitrary term which must be introduced into the solution of the equation

$$V\nabla\sigma = \tau.$$

To solve this equation is (§ 384) to find the displacement of any one of a group of points when the consequent rotation is given.

Here $$S\nabla\tau = S.\nabla V\nabla\sigma = S\nabla^2\sigma = 0;$$

so that, omitting the arbitrary term (§ 493), we have

$$\nabla^2\sigma = \nabla\tau,$$

and each constituent of $\sigma$ is, as above, determinate. Compare § 503.

Thomson* has put the solution in a form which may be written

$$\sigma = \tfrac{1}{3}\int V\tau\mathbf{d}\rho + \nabla u,$$

if we understand by $\int(\quad)\,\mathbf{d}\rho$ integrating the term in $\mathbf{d}x$ as if $y$ and $z$ were constants, &c. Bearing this in mind, we have as verification,

$$V\nabla\sigma = \tfrac{1}{3}\Sigma\, Vi\left\{V\tau i + \int V\frac{d\tau}{dx}\,\mathbf{d}\rho\right\}$$

$$= \tfrac{1}{3}\left\{2\tau + \Sigma\int\frac{d\tau}{dx}\,\mathbf{d}x + \Sigma\int\mathbf{d}\rho Si\frac{d\tau}{dx}\right\}$$

$$= \tfrac{1}{3}\{3\tau + \int d\rho S\nabla\tau\} = \tau.$$

495. We now come to relations between the results of integration extended over a non-closed surface and round its boundary.

Let $\sigma$ be any vector function of the position of a point. The line-integral whose value we seek as a fundamental theorem is

$$\int S\sigma d\tau,$$

where $\tau$ is the vector of any point in a small closed curve, drawn from a point within it, and in its plane.

---

* *Electrostatics and Magnetism*, § 521, or *Phil. Trans.*, 1852.

Let $\sigma_0$ be the value of $\sigma$ at the origin of $\tau$, then

$$\sigma = \sigma_0 - S(\tau\nabla)\sigma_0,$$

so that
$$\int S\sigma d\tau = \int S.\{\sigma_0 - S(\tau\nabla)\sigma_0\} d\tau.$$

But
$$\int d\tau = 0,$$

because the curve is closed; and (anté, § 467) we have generally

$$\int S\tau\nabla S\sigma_0 d\tau = \tfrac{1}{2} S\nabla(\tau S\sigma_0\tau - \sigma_0 \int V\tau d\tau).$$

Here the integrated part vanishes for a closed circuit, and

$$\tfrac{1}{2}\int V\tau d\tau = ds U\nu,$$

where $ds$ is the area of the small closed curve, and $U\nu$ is a unit-vector perpendicular to its plane. Hence

$$\int S\sigma_0 d\tau = S.\nabla\sigma_0 U\nu.ds.$$

Now, any finite portion of a surface may be broken up into small elements such as we have just treated, and the sign only of the integral along each portion of a bounding curve is changed when we go round it in the opposite direction. Hence, just as Ampère did with electric currents, substituting for a finite closed circuit a network of an infinite number of infinitely small ones, in each contiguous pair of which the common boundary is described by equal currents in opposite directions, we have for a finite unclosed surface

$$\int S\sigma d\rho = \iint S.\nabla\sigma U\nu.ds.$$

There is no difficulty in extending this result to cases in which the bounding curve consists of detached ovals, or possesses multiple points.

This theorem seems to have been first given by Stokes (*Smith's Prize Exam.* 1854), in the form

$$\int(\alpha dx + \beta dy + \gamma dz)$$
$$= \iint ds \left\{ l\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) + m\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) + n\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)\right\}.$$

It solves the problem suggested by the result of § 490 above. It will be shewn, however, in a later section that the equation above, though apparently quite different from that of § 484, is merely a particular case of it.

[If we recur to the case of an infinitely small area, it is clear that

$$\iint S.\nabla\sigma U\nu ds$$

is a maximum when

$$V.U\nu V\nabla\sigma = 0.$$

Hence $V\nabla\sigma$ is, at every point, perpendicular to a small area for which

$$\int S\sigma d\rho$$

is a maximum.]

496. If $\sigma$ represent the vector force acting on a particle of matter at $\rho$, $-S.\sigma d\rho$ represents the work done by it while the particle is displaced along $d\rho$, so that the single integral

$$\int S\sigma d\rho$$

of last section, taken with a negative sign, represents the work done during a complete cycle. When this integral vanishes it is evident that, if the path be divided into any two parts, the work spent during the particle's motion through one part is equal to that gained in the other. Hence the system of forces must be conservative, i.e., must do the same amount of work for all paths having the same extremities.

But the equivalent double integral must also vanish. Hence a conservative system is such that

$$\iint ds S.\nabla\sigma U\nu = 0,$$

whatever be the form of the finite portion of surface of which $ds$ is an element. Hence, as $\nabla\sigma$ has a fixed value at each point of space, while $U\nu$ may be altered at will, we must have

$$V\nabla\sigma = 0,$$

or

$$\nabla\sigma = \text{scalar}.$$

If we call $X$, $Y$, $Z$ the component forces parallel to rectangular axes, this extremely simple equation is equivalent to the well-known conditions

$$\frac{dX}{dy} - \frac{dY}{dx} = 0, \quad \frac{dY}{dz} - \frac{dZ}{dy} = 0, \quad \frac{dZ}{dx} - \frac{dX}{dz} = 0.$$

Returning to the quaternion form, as far less complex, we see that

$$\nabla\sigma = \text{scalar} = 4\pi r, \text{ suppose,}$$

implies that

$$\sigma = \nabla P,$$

where $P$ is a scalar such that

$$\nabla^2 P = 4\pi r;$$

that is, $P$ is the potential of a distribution of matter, magnetism, or statical electricity, of volume-density $r$.

Hence, for a non-closed path, under conservative forces,

$$-\int S.\sigma d\rho = -\int S.\nabla P d\rho = -\int S(d\rho\nabla) P$$
$$= \int d_{d\rho} P = \int dP = P_2 - P_1,$$

depending solely on the values of $P$ at the extremities of the path.

497. A vector theorem, which is of great use, and which corresponds to the scalar theorem of § 491, may easily be obtained. Thus, with the notation already employed,

$$\int V.\sigma d\tau = \int V\{\sigma_0 - S(\tau\nabla)\sigma_0\} d\tau,$$
$$= -\int S(\tau\nabla) V.\sigma_0 d\tau.$$

Now $\quad V(V.\nabla V.\tau d\tau)\sigma_0 = -S(\tau\nabla) V.\sigma_0 d\tau - S(d\tau\nabla) V\tau\sigma_0,$

and $\quad d\{S(\tau\nabla) V\sigma_0\tau\} = S(\tau\nabla) V.\sigma_0 d\tau + S(d\tau\nabla) V\sigma_0\tau.$

Subtracting, and omitting the term which is the same at both limits, we have

$$\int V.\sigma d\tau = -V.(VU\nu\nabla)\sigma_0 ds.$$

Extended as above to any closed curve, this takes at once the form

$$\int V.\sigma d\rho = -\iint ds V.(VU\nu\nabla)\sigma.$$

Of course, in many cases of the attempted representation of a quaternion surface-integral by another taken round its bounding curve, we are met by ambiguities as in the case of the space-integral, § 492: but their origin, both analytically and physically, is in general obvious.

498. The following short investigation gives, in a complete form, the kernel of the whole of this part of the subject. But §§ 495—7 have still some interest of their own.

If $P$ be any scalar function of $\rho$, we have (by the process of § 495, above)

$$\int P d\tau = \int\{P_0 - S(\tau\nabla) P_0\} d\tau$$
$$= -\int S.\tau\nabla P_0.d\tau.$$

But $\quad V.\nabla V.\tau d\tau = d\tau S.\tau\nabla - \tau S.d\tau\nabla,$

and $\quad d(\tau S\tau\nabla) = d\tau S.\tau\nabla + \tau S.d\tau\nabla.$

These give

$$\int P d\tau = -\tfrac{1}{2}\{\tau S\tau\nabla - V.\int V(\tau d\tau)\nabla\} P_0 = ds V.U\nu\nabla P_0.$$

Hence, for a closed curve of any form, we have

$$\int P d\rho = \iint ds V.U\nu\nabla P,$$

from which the theorems of §§ 495, 497 may easily be deduced.

Multiply into any constant vector, and we have, by adding three such results

$$\int d\rho\sigma = \iint ds\, V(U\nu\nabla)\,\sigma. \quad \text{[See § 497.]}$$

Hence at once (by adding together the corresponding members of the two last equations, and putting

$$P + \sigma = q)$$

$$\int d\rho q = \iint ds\, V(U\nu\nabla)\, q,$$

where $q$ is any quaternion whatever.

[For the reason why we have no corresponding formula, with $S$ instead of $V$ in the right-hand member, see remark in [ ] in § 505.]

499. Commencing afresh with the fundamental integral

$$\iiint S\nabla\sigma d\varsigma = \iint S\sigma U\nu ds,$$

put

$$\sigma = u\beta,$$

and we have

$$\iiint S\beta\nabla u d\varsigma = \iint u S\beta U\nu ds;$$

from which at once

$$\iiint \nabla u d\varsigma = \iint u U\nu ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$\iiint \nabla\tau d\varsigma = \iint U\nu\,.\,\tau ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which gives

$$\iiint V\nabla V\sigma\tau d\varsigma = V\iint U\nu V\sigma\tau ds = \iint (\tau SU\nu\sigma - \sigma SU\nu\tau)\, ds.$$

Equation (2) gives a remarkable expression for the surface of a space in terms of a volume-integral. For take

$$\tau = U\nu = U\nabla P,$$

where

$$P = \text{const.}$$

is the scalar equation of the closed bounding surface. Then

$$-\iint ds = \iint U\nu U\nu ds = \iiint \nabla U\nabla P d\varsigma.$$

(Note that this implies

$$\iiint V\nabla U\nabla P d\varsigma = 0,$$

which in itself is remarkable.)

Thus the surface of an ellipsoid

$$S\rho\phi\rho = C,$$

is

$$\iiint \nabla U\phi\rho d\varsigma,$$

the integration being carried on throughout the enclosed space. (Compare § 485.)

Again, in (2), putting $u_1\tau$ for $\tau$, and taking the scalar, we have

$$\iiint (S\tau\nabla u_1 + u_1 S\nabla\tau)\, d\varsigma = \iint u_1 S\tau U\nu\, ds,$$

whence $$\iiint \{S(\tau\nabla)\,\sigma + \sigma S\nabla\tau\}\, d\varsigma = \iint \sigma S\tau U\nu ds \qquad \text{.......... ...(3).}$$

The sum of (1) and (2) gives, for any quaternion,

$$\iiint \nabla q d\varsigma = \iint U\nu \,.\, q ds.$$

The final formulae in this, and in the preceding, section give expressions in terms of surface integrals for the volume, and the line, integrals of a quaternion. The latter is perfectly general, but (for a reason pointed out in § 492) the former is definite only when the quaternion has the form $\nabla q$.

500. The fundamental form of the Volume and Surface Integral is (as in § 499 (1))

$$\iiint \nabla u d\varsigma = \iint U\nu u ds.$$

Apply it to a space consisting of a very thin transverse slice of a cylinder. Let $t$ be the thickness of the slice, $A$ the area of one end, and $\alpha$ a unit-vector perpendicular to the plane of the end. The above equation gives at once

$$V(\alpha\nabla)\, u \,.\, tA = t\int V\alpha U\nu \,.\, u dl,$$

where $dl$ is the length of an element of the bounding curve of the section, and the only values of $U\nu$ left are parallel to the plane of the section and normal to the bounding curve. If we now put $\rho$ as the vector of a point in that curve, it is plain that

$$V.\,\alpha U\nu = U d\rho, \quad dl = T d\rho,$$

and the expression becomes

$$V(\alpha\nabla)\, u \,.\, A = \int u d\rho.$$

By juxtaposition of an infinite number of these infinitely small directed elements, $\alpha$ (now to be called $U\nu$) being the normal vector of the area $A$ (now to be called $ds$), we have at once

$$\iint V(U\nu\nabla)\, u ds = \int u d\rho,$$

which is the fundamental form of the Surface and Line Integral, as given in § 498. Hence, as stated in § 484, these relations are not independent.

In fact, as the first of these expressions can be derived at once from the ordinary equation of "continuity," so the second is merely the particular case corresponding to displacements confined to a given surface. It is left to the student to obtain it, simply and directly, (in the form of § 495) from this consideration.

[*Note.* A remark of some importance must be made here. It may be asked:—Why not adopt for the proof of the fundamental theorems of the present subject the obvious Newtonian process (as applied, for instance, in Thomson and Tait's *Natural Philosophy*, § 194, or in Clerk-Maxwell's *Electricity*, § 591)?

The reply is that, while one great object of the present work is (as far as possible) to banish artifice, and to shew the "perfect naturalness of Quaternions," the chief merit of the beautiful process alluded to is that it forms one of the most intensely artificial applications of an essentially artificial system. Cartesian and Semi-Cartesian methods may be compared to a primitive telegraphic code, in which the different signals are assigned to the various letters at hap-hazard; Quaternions to the natural system, in which the simplest signals are reserved for the most frequently recurring letters. In the former system some one word, or even sentence, may occasionally be more simply expressed than in the latter:—though there can be no doubt as to which system is to be preferred. But, even were it not so, the methods we have adopted in the present case give a truly marvellous insight into the real meaning and "inner nature" of the formulae obtained.]

501. As another example of the important results derived from the simple formulae of § 499, take the following, viz.:—

$$\iint V.V(\sigma U\nu)\,\tau ds = \iint \sigma S\tau U\nu ds - \iint U\nu S\sigma\tau ds,$$

where by (3) and (1) of that section we see that the right-hand member may be written

$$= \iiint \{S(\tau\nabla)\,\sigma + \sigma S\nabla\tau - \nabla S\sigma\tau\}\, d\varsigma$$

$$= -\iiint V.V(\nabla\sigma)\,\tau d\varsigma \text{..............................}(4).$$

In this expression the student must remark that $\nabla$ operates on $\tau$ as well as on $\sigma$. Had it operated on $\tau$ alone, we should have inverted the order of $\nabla$ and $\sigma$, and changed the sign of the whole; or we might have had recourse to the notation of the end of § 133.

This, and similar formulae, are easily applied to find the potential and the vector-force due to various distributions of magnetism. To shew how they are to be introduced, we briefly sketch the mode of expressing the potential of a distribution.

## *K. Application of the ∇ Integrals to Magnetic &c. Problems.*

502. Let $\sigma$ be the vector expressing the direction and intensity of magnetisation, per unit of volume, at the element $d\varsigma$. Then if the magnet be placed in a field of magnetic force whose potential is $u$, we have for its potential energy

$$\begin{aligned} E &= -\iiint S\sigma\nabla u d\varsigma \\ &= \iiint uS\nabla\sigma d\varsigma - \iiint S\nabla(u\sigma)\, d\varsigma \\ &= \iiint uS\nabla\sigma d\varsigma - \iint uS\sigma U\nu ds. \end{aligned}$$

This shews at once that the magnetism may be resolved into a volume-density $S\nabla\sigma$, and a surface-density $-S\sigma U\nu$. Hence, for a solenoidal distribution,

$$S\nabla\sigma = 0.$$

What Thomson has called a lamellar distribution (*Phil. Trans.* 1852), obviously requires that

$$S\sigma d\rho$$

be integrable without a factor; i.e., that

$$V\nabla\sigma = 0.$$

A complex lamellar distribution requires that the same expression be integrable by the aid of a factor. If this be $u$, we have at once

$$V\nabla(u\sigma) = 0,$$

or

$$S.\sigma\nabla\sigma = 0.$$

But we easily see that (4) of § 501 may be written

$$-\iint V.(V\sigma U\nu)\tau ds = -\iiint V.\tau V\nabla\sigma d\varsigma - \iiint V.\sigma\nabla\tau d\varsigma + \iiint S\sigma\nabla.\tau d\varsigma.$$

Now, if

$$\tau = \nabla\left(\frac{1}{r}\right),$$

where $r$ is the distance between any external point and the element $d\varsigma$, the last term on the right is the vector-force exerted by the magnet on a unit-pole placed at the point. The second term on the right vanishes by Laplace's equation, and the first vanishes as above if the distribution of magnetism be lamellar, thus giving Thomson's result in the form of a surface integral.

503. As another instance, let $\alpha$ be the vector of magnetic induction, $\beta$ the vector potential, at any point. Then we know, physically, that

$$\int S\beta d\rho = \iint SU\nu\alpha.ds.$$

But, by the theorem of § 496, we have

$$\int S\beta d\rho = \iint S \,.\, U\nu\nabla\beta ds.$$

Since the boundary and the enclosed surface may be any whatever, we must have

$$\alpha = V\nabla\beta;$$

and, as a consequence,

$$S\nabla\alpha = 0.$$

Hence $$\nabla\beta = \alpha + u,$$

$$\beta = \nabla^{-1}\alpha + \nabla^{-1}u + \nabla q,$$

where $q$ is a quaternion satisfying the equations

$$\nabla^2 q = 0, \qquad S\nabla q = 0.$$

To interpret the other terms, let

$$\nabla^{-1}u = \sigma,$$

so that $$V\nabla\sigma = 0;$$

and $$\sigma = \nabla v,$$

where $v$ is a scalar such that

$$\nabla^2 v = u.$$

Thus $v$ is the potential of a distribution $u/4\pi$, and can therefore be found without ambiguity when $u$ is given. And of course

$$\nabla^{-1}u = \nabla v$$

is also found without ambiguity.

Again, as $$S\nabla\alpha = 0,$$

we have $$\nabla\alpha = \gamma, \text{ suppose.}$$

Hence, for any assumed value of $\gamma$, we have

$$\nabla^{-1}\alpha = \nabla^{-2}\gamma,$$

so that this term of the above value of $\beta$ is also found without ambiguity.

The auxiliary quaternion $q$ depends upon potentials of arbitrary distributions wholly outside the space to which the investigation may be limited. [Compare this section with § 494.]

504. An application may be made of similar transformations to Ampère's *Directrice de l'action électrodynamique*, which, § 458 above, is the vector-integral

$$\int \frac{V\rho d\rho}{T\rho^3},$$

where $d\rho$ is an element of a closed circuit, and the integration extends round the circuit. This may be written

$$-\int V d\rho \nabla \frac{1}{r},$$

so that its value as a surface integral is

$$\iint S(U\nu\nabla)\nabla\frac{1}{r}ds - \iint U\nu\nabla^2\frac{1}{r}ds.$$

Of this the last term vanishes, unless the origin is in, or infinitely near to, the surface over which the double integration extends. The value of the first term is seen (by what precedes) to be the vector-force due to uniform normal magnetisation of the same surface. Thus we see the reason why the *Directrice* can be expressed in terms of the spherical opening of the circuit, as in §§ 459, 471.

505. The following result is obviously but one of an extensive class of useful transformations. Since

$$\nabla U\rho = -\frac{2}{T\rho},$$

we obtain at once from § 484

$$-2\iiint \frac{d\varsigma}{T\rho} = \iint SU\rho U\nu ds,$$

a curious expression for the gravitation potential of a homogeneous body in terms of a surface-integral.

[The right-hand member may be written as

$$\iint SU\nu\nabla T\rho ds;$$

and an examination into its nature shews us *why* we ought not to expect to have a general expression for

$$\iint SU\nu\nabla P ds$$

in terms of a line-integral. It will be excellent practice for the student to make this examination himself. Of course, a more general method presents itself in finding the volume-integral which is equivalent to the last written surface-integral extended to the surface of a closed space.]

From this, by differentiation with respect to $\alpha$, after putting $\rho+\alpha$ for $\rho$, or by expanding in ascending powers of $T\alpha$ (both of which tacitly assume that the origin is external to the space integrated through, i.e., that $T\rho$ nowhere vanishes within the limits), we have

$$-2\iiint\frac{d\varsigma U\rho}{T\rho^2}=\iint\frac{V.U\rho V.U\nu U\rho}{T\rho}ds=2\iint\frac{U\nu ds}{T\rho};$$

and this, again, involves

$$\iint\frac{U\nu ds}{T\rho}=\iint\frac{U\rho}{T\rho}SU\nu U\rho ds.$$

506. The interpretation of these, and of more complex formulae of a similar kind, leads to many curious theorems in attraction and in potentials. Thus, from (1) of § 499, we have

$$\iiint\frac{\nabla t}{T\rho}d\varsigma-\iiint\frac{tU\rho}{T\rho^2}d\varsigma=\iint\frac{tU\nu}{T\rho}ds\ldots\ldots\ldots\ldots\ldots(1),$$

which gives the attraction of a mass of density $t$ in terms of the potentials of volume distributions and surface distributions. Putting

$$\sigma=it_1+jt_2+kt_3,$$

this becomes

$$\iiint\frac{\nabla\sigma d\varsigma}{T\rho}-\iiint\frac{U\rho.\sigma d\varsigma}{T\rho^2}=\iint\frac{U\nu.\sigma ds}{T\rho}.$$

By putting $\sigma=\rho$, and taking the scalar, we recover a formula given above; and by taking the vector we have

$$V\iint U\nu U\rho ds=0.$$

This may be easily verified from the formula

$$\int Pd\rho=V\iint U\nu.\nabla Pds,$$

by remembering that $\nabla T\rho=U\rho.$

Again if, in the fundamental integral, we put

$$\sigma=tU\rho,$$

we have $$\iiint\frac{S(\rho\nabla)t}{T\rho}d\varsigma-2\iiint\frac{td\varsigma}{T\rho}=\iint tSU\nu U\rho ds.$$

[It is curious how closely, in fact to a numerical factor of one term *près*, this equation resembles what we should get by operating on (1) by $S.\rho$, and supposing that we could put $\rho$ under the signs of integration.]

### L. *Application of $\nabla$ to the Stress-Function.*

507. As another application, let us consider briefly the *Stress-function in an elastic solid.*

At any point of a strained body let $\lambda$ be the vector stress per unit of area perpendicular to $i$, $\mu$ and $\nu$ the same for planes perpendicular to $j$ and $k$ respectively.

Then, by considering an indefinitely small tetrahedron, we have for the stress per unit of area perpendicular to a unit-vector $\omega$ the expression

$$\lambda Si\omega + \mu Sj\omega + \nu Sk\omega = -\phi\omega,$$

so that the stress across any plane is represented by *a linear and vector function* of the unit normal to the plane.

But if we consider the equilibrium, as regards rotation, of an infinitely small parallelepiped whose edges are parallel to $i$, $j$, $k$ respectively, we have (supposing there are no molecular couples)

$$V(i\lambda + j\mu + k\nu) = 0,$$

or

$$\Sigma Vi\phi i = 0,$$

or, supposing $\nabla$ to apply to $\rho$ alone,

$$V.\nabla\phi\rho = 0.$$

This shews (§ 185) that in the present case $\phi$ is *self-conjugate*, and thus involves not nine distinct constants but only six.

508. Consider next the equilibrium, as regards translation, of any portion of the solid filling a simply-connected closed space. Let $u$ be the potential of the external forces. Then the condition is obviously

$$\iint\phi(U\nu)\,ds + \iiint d\varsigma\nabla u = 0,$$

where $\nu$ is the normal vector of the element of surface $ds$. Here the double integral extends over the whole boundary of the closed space, and the triple integral throughout the whole interior.

To reduce this to a form to which the method of § 485 is directly applicable, operate by $S.\alpha$ where $\alpha$ is any constant vector whatever, and we have

$$\iint S.\phi\alpha U\nu ds + \iiint d\varsigma S\alpha\nabla u = 0$$

by taking advantage of the self-conjugateness of $\phi$. This may be written (by transforming the surface-integral into a volume-integral)

$$\iiint d\varsigma\,(S.\nabla\phi\alpha + S.\alpha\nabla u) = 0,$$

and, as the limits of integration may be any whatever,

$$S.\nabla\phi\alpha + S\alpha\nabla u = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is the required equation, the indeterminateness of $\alpha$ rendering it equivalent to *three* scalar conditions.

There are various modes of expressing this without the $\alpha$. Thus, if $\Delta$ be used for $\nabla$ when the constituents of $\phi$ are considered, we may write

$$\nabla u = S\nabla\Delta.\phi\rho.$$

It is easy to see that the right-hand member may be put in either of the equivalent forms

$$\nabla S.\Delta\phi\rho \quad \text{or} \quad S.\Delta\phi\nabla.\rho.$$

In integrating this expression through a given space, we must remark that $\nabla$ and $\rho$ are merely temporary symbols of construction, and therefore are not to be looked on as variables in the integral.

Instead of transforming the surface-integral, we might have begun by transforming the volume-integral. Thus the first equation of this section gives

$$\iint(\phi + u)\,U\nu ds = 0.$$

From this we have at once

$$\iint S.U\nu\,(\phi + u)\,\alpha ds = 0.$$

Thus, by the result of § 490, whatever be $\alpha$ we have

$$S.\nabla(\phi + u)\,\alpha = 0,$$

which is the condition obtained by the former process.

As a verification, it may perhaps be well to shew that from this equation we can get the condition of equilibrium, as regards *rotation*, of a simply-connected portion of the body, which can be written by inspection as

$$\iint V.\rho\phi\,(U\nu)\,ds + \iiint V\rho\nabla u d\varsigma = 0.$$

This is easily done as follows: (1) gives

$$S.\nabla\phi\sigma + S\sigma\nabla u = 0,$$

if, and only if, $\sigma$ satisfy the condition

$$S.\phi(\nabla)\,\sigma = 0.$$

Now this condition is satisfied by

$$\sigma = V\alpha\rho$$

where $\alpha$ is any constant vector. For

$$S.\phi(\nabla)\,V\alpha\rho = -S.\alpha V\phi(\nabla)\,\rho = S.\alpha V\nabla_1\phi\rho_1 = 0,$$

in consequence of the self-conjugateness of $\phi$. Hence

$$\iiint d\varsigma\,(S.\nabla\phi V\alpha\rho + S.\alpha\rho\nabla u) = 0,$$

or

$$\iint ds S.\alpha\rho\phi U\nu + \iiint d\varsigma S.\alpha\rho\nabla u = 0.$$

Multiplying by $\alpha$, and adding the results obtained by making $\alpha$ in succession each of three rectangular vectors, we obtain the required equation.

509. *To find the stress-function in terms of the displacement at each point of an isotropic solid,* when the resulting strain is small, we may conveniently apply the approximate method of § 384. As the displacement is supposed to be continuous, the strain in the immediate neighbourhood of any point may be treated as homogeneous. Thus, round each point, there is one series of rectangular parallelepipeds, each of which remains rectangular after the strain. Let $\alpha, \beta, \gamma$; $\alpha_1, \beta_1, \gamma_1$; be unit vectors parallel to their edges before, and after, the strain respectively; and let $e_1, e_2, e_3$ be the elongations of unit edges parallel to these lines. We shall not have occasion to determine these quantities, as they will be eliminated after having served to form the requisite equations.

Since the solid is isotropic and homogeneous, the stress is perpendicular to each face in the strained parallelepipeds; and its amount (per unit area) can be expressed as

$$P_1 = 2ne_1 + (c - \tfrac{2}{3}n)\,\Sigma e, \text{ \&c.} \qquad\qquad (1),$$

where $n$ and $c$ are, respectively, the rigidity and the resistance to compression.

Next, as in § 384, let $\sigma$ be the displacement at $\rho$. The strain-function is

$$\psi\varpi = \varpi - S\varpi\nabla.\sigma$$

so that at once

$$\Sigma e = -S\nabla\sigma; \qquad\qquad (2),$$

and, if $q(\ \ )q^{-1}$ be the operator which turns $\alpha$ into $\alpha_1$, &c., we have

$$q\varpi q^{-1} = \varpi + \tfrac{1}{2}V.V(\nabla\sigma)\,\varpi \qquad\qquad (3).$$

Thus, if $\phi$ be the stress-function, we have (as in § 507)

$$\phi\omega = -\Sigma . P_1\alpha_1 S\alpha_1\omega \text{.........................} (4).$$

But $$\psi\omega = -\Sigma . (1+e_1)\,\alpha_1 S\alpha\omega,$$

so that $$\psi'\omega = -\Sigma . (1+e_1)\,\alpha S\alpha_1\omega,$$

and $$q\psi'\omega q^{-1} = -\Sigma . (1+e_1)\,\alpha_1 S\alpha_1\omega \text{ ..................} (5).$$

By the help of (1), (2), and (5), (4) becomes

$$\phi\omega = 2nq\psi'\omega q^{-1} - 2n\omega - (c - \tfrac{2}{3}n)\,\omega S\nabla\sigma\,;$$

and, to the degree of approximation employed, (3) shews that this may be written

$$\phi\omega = -n\,(S\omega\nabla . \sigma + \nabla S\omega\sigma) - (c - \tfrac{2}{3}n)\,\omega S\nabla\sigma \text{.........} (6),$$

which is the required expression, the function $\phi$ being obviously self-conjugate.

As an example of its use, suppose the strain to be a uniform dilatation. Here

$$\sigma = e\rho,$$

and $$\phi\omega = 2ne\omega + 3\,(c - \tfrac{2}{3}n)\,e\omega = 3ce\omega\,;$$

denoting traction $3ce$, uniform in all directions. If $e$ be negative, there is uniform condensation, and the stress is simply hydrostatic pressure.

Again, let $$\sigma = -e\alpha S\alpha\rho,$$

which denotes uniform extension in one direction, unaccompanied by transversal displacement. We have ($\alpha$ being a unit vector)

$$\phi\omega = -2ne\alpha S\omega\alpha + (c - \tfrac{2}{3}n)\,e\omega.$$

Thus along $\alpha$ there is traction

$$(c + \tfrac{4}{3}n)e,$$

but in all directions perpendicular to $\alpha$ there is also traction

$$(c - \tfrac{2}{3}n)\,e.$$

Finally, take the displacement

$$\sigma = -e\alpha S\beta\rho.$$

It gives $$\phi\omega = -ne\,(\alpha S\omega\beta + \beta S\omega\alpha) - (c - \tfrac{2}{3}n)\,e\omega S\alpha\beta.$$

This displacement gives a simple shear if the unit vectors $\alpha$ and $\beta$ are at right angles to one another, and then

$$\phi\omega = -ne\,(\alpha S\omega\beta + \beta S\omega\alpha),$$

which agrees with the well-known results. In particular, it shews that the stress is wholly tangential on planes perpendicular either to $\alpha$ or to $\beta$; and wholly normal on planes equally inclined to them and perpendicular to their plane. The symmetry shews that the stress will not be affected by interchange of the unit vectors, $\alpha$ and $\beta$, in the expression for the displacement.

510. The work done by the stress on any simply connected portion of the solid is obviously

$$W = \tfrac{1}{2} \iint S \,.\, \phi\,(U\nu)\, \sigma ds,$$

because $\phi\,(U\nu)$ is the vector force overcome per unit of area on the element $ds$. [The displacement at any moment may be written $x\sigma$; and, as the stress is always proportional to the strain, the factor $xdx$ has to be integrated from 0 to 1.] This is easily transformed to

$$W = \tfrac{1}{2} \iiint S \,.\, \nabla \phi \sigma d\varsigma.$$

511. We may easily obtain the general expression for the work corresponding to a strain in any elastic solid. The physical principles on which we proceed are those explained in *Appendix C* to Thomson and Tait's *Natural Philosophy*. The mode in which they are introduced, however, is entirely different; and a comparison will shew the superiority of the Quaternion notation, alike in compactness and in intelligibility and suggestiveness.

If the strain, due to the displacement $\sigma$, viz.

$$\psi\tau = \tau - S\tau\nabla \,.\, \sigma$$

be a mere rotation, in which case of course no work is stored up by the stress, we have at once

$$S \,.\, \psi\omega\psi\tau = S\omega\tau$$

for all values of $\omega$ and $\tau$. We may write this as

$$S \,.\, \omega\,(\psi'\psi - 1)\,\tau = S\omega\chi\tau = 0,$$

where $\chi$ is (§ 380) a self-conjugate linear and vector function, whose complete value is

$$\chi\tau = -S\tau\nabla \,.\, \sigma - \nabla S\tau\sigma + \nabla_1 S\tau\nabla S\sigma\sigma_1.$$

The last term of this may, in many cases, be neglected.

When the strain is very small, the work (per unit volume) must thus obviously be a homogeneous function, of the second degree, of the various independent values of the expression

$$S\omega\chi\tau.$$

On account of the self-conjugateness of $\chi$ there are but *six* such values:—viz.

$$Si\chi i,\ Si\chi j,\ Si\chi k,\ Sj\chi j,\ Sj\chi k,\ Sk\chi k.$$

Their homogeneous products of the second degree are therefore *twenty-one* in number, and this is the number of elastic coefficients which must appear in the general expression for the work. In the most general form of the problem these coefficients are to be regarded as given functions of $\rho$.

At and near any one point of the body, however, we may take $i, j, k$ as the chief vectors of $\chi$ at that point, and then the work for a small element is expressible in terms of the six homogeneous products, of the second degree, of the three quantities

$$Si\chi i,\ Sj\chi j,\ Sk\chi k.$$

This statement will of course extend to a portion of the body of any size if (whether isotropic or not) it be homogeneous and homogeneously strained. From this follow at once all the elementary properties of homogeneous stress.

### *M. The Hydrokinetic Equations.*

512. As another application, let us form the hydrokinetic equations, on the hypothesis that a perfect fluid is *not* a molecular assemblage but a continuous medium.

Let $\sigma$ be the vector-velocity of a very small part of the fluid at $\rho$; $e$ the density there, taken to be a function of the pressure, $p$, alone; i.e. supposing that the fluid is homogeneous when the pressure is the same throughout; $P$ the potential energy of unit mass at the point $\rho$.

The equation of "continuity" is to be found by expressing the fact that the increase of mass in a small fixed space is equal to the excess of the fluid which has entered over that which has escaped. If we take the volume of this space as unit, the condition is

$$\frac{de}{dt} = \iint SU\nu\,(e\sigma)\,ds = S\nabla\,(e\sigma) \qquad\qquad (1).$$

We may put this, if we please, in the form

$$\frac{\partial e}{\partial t} = \frac{de}{dt} - S\sigma\nabla\,.\,e = eS\nabla\sigma \qquad\qquad (2),$$

where $\partial$ expresses total differentiation, or, in other words, that we follow a definite portion of the fluid in its motion.

The expression might at once have been written in the form (2) from the comparison of the results of two different methods of representing the rate of increase of density of a small portion of the fluid as it moves along. Both forms reduce to

$$S\nabla\sigma = 0,$$

when there is no change of density (§ 384).

Similarly, for the rate of increase of the whole momentum within the fixed unit space, we have

$$\frac{d(e\sigma)}{dt} = -e\nabla P - \nabla p + \iint SU\nu\sigma \,.\, e\sigma ds\,;$$

where the meanings of the first two terms are obvious, and the third is the excess of momentum of the fluid which enters, over that of the fluid which leaves, the unit space.

The value of the double integral is, by § 499 (3),

$$\sigma S\nabla(e\sigma) + eS\sigma\nabla \,.\, \sigma = \sigma\frac{de}{dt} + eS\sigma\nabla \,.\, \sigma, \text{ by (1).}$$

Thus we have, for the equation of motion,

$$e\frac{\partial\sigma}{\partial t} = e\frac{d\sigma}{dt} - eS\sigma\nabla \,.\, \sigma = -e\nabla P - \nabla p\,;$$

or, finally

$$\frac{\partial\sigma}{\partial t} = -\nabla\left(P + \int\frac{dp}{e}\right) = -\nabla Q \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This, in its turn, might have been even more easily obtained by dealing with a small definite portion of the fluid.

It is necessary to observe that in what precedes we have tacitly assumed that $\sigma$ is continuous throughout the part of the fluid to which the investigation applies :—i.e. that there is neither rupture nor finite sliding.

513. There are many ways of dealing with the equation (3) of fluid motion. We select a few of those which, while of historic interest, best illustrate quaternion methods.

We may write (3) as

$$\frac{d\sigma}{dt} - S\sigma\nabla \,.\, \sigma = -\nabla Q.$$

Now we have always

$$V \,.\, \sigma V\nabla\sigma = S\sigma\nabla \,.\, \sigma - \nabla_1 S\sigma_1\sigma = S\sigma\nabla \,.\, \sigma - \tfrac{1}{2}\nabla \,.\, \sigma^2.$$

Hence, if the motion be irrotational, so that (§ 384)

$$V\nabla\sigma = 0,$$

the equation becomes

$$\frac{d\sigma}{dt} - \tfrac{1}{2}\nabla . \sigma^2 = -\nabla Q.$$

But, if $w$ be the velocity-potential,

$$\sigma = \nabla w;$$

and we obtain (by substituting this in the first term, and operating on the whole by $S . d\rho$) the common form

$$d\frac{dw}{dt} + \tfrac{1}{2}d . v^2 = -dQ,$$

where $\frac{1}{2}v^2 (= -\frac{1}{2}\sigma^2)$ is the kinetic energy of unit mass of the fluid.

If the fluid be incompressible, we have Laplace's equation for $w$, viz.

$$\nabla^2 w = S\nabla\sigma = 0.$$

When there is no velocity-potential, we may adopt Helmholtz' method. But first note the following quaternion transformation (*Proc. R. S. E.* 1869—70)

$$\frac{\partial}{\partial t}\nabla\sigma - \nabla\frac{\partial}{\partial t}\sigma = \nabla_1 S\sigma_1\nabla . \sigma.$$

[The expression on the right has many remarkable forms, the finding of which we leave, as an exercise, to the student. For our present purpose it is sufficient to know that its vector part is

$$-S . \nabla_1\sigma_1\nabla . \sigma + V\nabla\sigma . S\nabla\sigma.]$$

This premised, operate on (3) by $V . \nabla$, and we have

$$V\nabla\frac{\partial\sigma}{\partial t} = 0.$$

Hence at once, if the fluid be compressible,

$$\frac{\partial}{\partial t}V\nabla\sigma = -S . \nabla_1\sigma_1\nabla . \sigma + V\nabla\sigma . S\nabla\sigma = V . \nabla V . \sigma V\nabla_1\sigma_1.$$

But if the fluid be incompressible

$$\frac{\partial}{\partial t}V\nabla\sigma = -S . \nabla_1\sigma_1\nabla . \sigma.$$

Either form shews that when the vector-rotation vanishes, its rate of change also vanishes. In other words, those elements of the fluid, which were originally devoid of rotation, remain so during the motion.

Thomson's mode of dealing with (3) is to introduce the integral

$$f = -\int_{\alpha}^{\rho} S\sigma d\rho \quad \text{.........................(5)}$$

which he calls the "flow" along the arc from the point $\alpha$ to the point $\rho$; these being points which move with the fluid.

Operating on (3) with $S\,.\,d\rho$, we have as above

$$\frac{\partial}{\partial t} S\sigma d\rho - S\sigma d\sigma = dQ\,;$$

so that, integrating along any definite line in the fluid from $\alpha$ to $\rho$, we have

$$-\frac{\partial f}{\partial t} = \left[ Q - \tfrac{1}{2}v^2 \right]_{\alpha}^{\rho} \quad \text{.........................(6)}$$

which gives the rate at which the flow along that line increases, as it swims along with the fluid.

If we integrate round a closed curve, the value of $\partial f/\partial t$ vanishes, because $Q$ is essentially a single-valued function. In this case the quantity $f$ is called the "circulation," and the result is stated in the form that the circulation round any definite path in the fluid retains a constant value.

Since the circulation is expressed by the complete integral

$$-\int S\sigma d\rho$$

it can also be expressed by the corresponding double integral

$$-\iint S\,.\,Uv\nabla\sigma ds,$$

so that it is only when there is at least one vortex-filament passing through the closed circuit that the circulation can have a finite value.

### *N. Use of ∇ in connection with Taylor's Theorem.*

514. Since the algebraic operator

$$\epsilon^{h\frac{d}{dx}},$$

when applied to any function of $x$, simply changes $x$ into $x+h$, it is obvious that if $\sigma$ be a vector not acted on by

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

we have

$$\epsilon^{-S\sigma\nabla} f(\rho) = f(\rho+\sigma),$$

whatever function $f$ may be. From this it is easy to deduce Taylor's theorem in one important quaternion form.

If $\Delta$ bear to the constituents of $\sigma$ the same relation as $\nabla$ bears to those of $\rho$, and if $f$ and $F$ be any two functions which satisfy the commutative law in multiplication, this theorem takes the curious form

$$\epsilon^{-S\Delta\nabla} f(\rho)\,F(\sigma) = f(\rho+\Delta)\,F(\sigma) = F(\sigma+\nabla)\,f(\rho);$$

of which a particular case is (in Cartesian symbols)

$$\epsilon^{\frac{d^2}{dx\,dy}} f(x)\,F(y) = f\left(x+\frac{d}{dy}\right)F(y) = F\left(y+\frac{d}{dx}\right)f(x).$$

The modifications which the general expression undergoes, when $f$ and $F$ are not commutative, are easily seen.

If one of these be an inverse function, such as, for instance, may occur in the solution of a linear differential equation, these theorems of course do not give the arbitrary part of the integral, but they often materially aid in the determination of the rest.

One of the chief uses of operators such as $S\alpha\nabla$, and various scalar functions of them, is to derive from $1/T\rho$ the various orders of Spherical Harmonics. This, however, is a very simple matter.

515. But there are among them results which appear startling from the excessively free use made of the separation of symbols. Of these one is quite sufficient to shew their general nature.

Let $P$ be any scalar function of $\rho$. It is required to find the difference between the value of $P$ at $\rho$, and its *mean* value throughout a very small sphere, of radius $r$ and volume $v$, which has the extremity of $\rho$ as centre. This, of course, can be answered at once from the formula of § 485. But the somewhat prolix method we are going to adopt is given for its own sake as a singular piece of analysis, not for the sake of the problem.

From what is said above, it is easy to see that we have the following expression for the required result:—

$$\frac{1}{v}\iiint(\epsilon^{-S\sigma\nabla}-1)\,P\,d\varsigma,$$

where $\sigma$ is the vector joining the centre of the sphere with the element of volume $d\varsigma$, and the integration (which relates to $\sigma$ and $d\varsigma$ alone) extends through the whole volume of the sphere.

Expanding the exponential, we may write this expression in the form

$$-\frac{1}{v}\iiint\left\{S\sigma\nabla-\frac{1}{2}(S\sigma\nabla)^2+\ldots\right\}Pd\varsigma$$

$$=-\frac{1}{v}S.\nabla P\iiint\sigma d\varsigma+\frac{1}{2v}\iiint(S\sigma\nabla)^2Pd\varsigma-\&c.,$$

higher terms being omitted on account of the smallness of $r$, the limit of $T\sigma$.

Now, symmetry shews at once that

$$\iiint\sigma d\varsigma=0.$$

Also, whatever constant vector be denoted by $\alpha$,

$$\iiint(S\alpha\sigma)^2d\varsigma=-\alpha^2\iiint(S\sigma U\alpha)^2d\varsigma.$$

Since the integration extends throughout a sphere, it is obvious that the integral on the right is half of what we may call the moment of inertia of the volume about a diameter. Hence

$$\iiint(S\sigma U\alpha)^2d\varsigma=\frac{vr^2}{5}.$$

If we now write $\nabla$ for $\alpha$, *as the integration does not refer to* $\nabla$, we have by the foregoing results (neglecting higher powers of $r$)

$$\frac{1}{v}\iiint(\epsilon^{-S\sigma\nabla}-1)Pd\varsigma=-\frac{r^2}{10}\nabla^2P,$$

which is the expression given by Clerk-Maxwell *. Although, for simplicity, $P$ has here been supposed a scalar, it is obvious that in the result above it may at once be written as a quaternion.

516. As another illustration, let us apply this process to the finding of the potential of a surface-distribution. If $\rho$ be the vector of the element $ds$, where the surface density is $f\rho$, the potential at $\sigma$ is

$$\iint dsf\rho\,.\,FT(\rho-\sigma),$$

$F$ being the potential function, which may have any form whatever.

By the preceding, § 514, this may be transformed into

$$\iint dsf\rho\,.\,\epsilon^{S\sigma\nabla}FT\rho;$$

or, far more conveniently for the integration, into

$$\iint dsf\rho\,.\,\epsilon^{S\rho\Delta}FT\sigma,$$

---

* *London Math. Soc. Proc.*, vol. iii, no. 34, 1871.

where $\Delta$ depends on the constituents of $\sigma$ in the same manner as $\nabla$ depends on those of $\rho$.

A still farther simplification may be introduced by using a vector $\sigma_0$, which is finally to be made zero, along with its corresponding operator $\Delta_0$, for the above expression then becomes

$$\iint ds \epsilon^{S\rho(\Delta-\Delta_0)} f\sigma_0 FT\sigma,$$

where $\rho$ appears in a comparatively manageable form. It is obvious that, so far, our formulae might be made applicable to any distribution. We now restrict them to a superficial one.

517. Integration of this last *form* can always be easily effected in the case of a surface of revolution, the origin being a point in the axis. For the expression, so far as the integration is concerned, can in that case be exhibited as a single integral

$$\int_p^q dx \mathsf{F}x\, \epsilon^{ax},$$

where $\mathsf{F}$ may be any scalar function, and $x$ depends on the cosine of the inclination of $\rho$ to the axis. And

$$\int_p^q dx \mathsf{F}x\, \epsilon^{ax} = \mathsf{F}\left(\frac{d}{da}\right) . \frac{\epsilon^{qa} - \epsilon^{pa}}{a}.$$

As the interpretation of the general results is a little troublesome, let us take the case of a spherical shell, the origin being the centre and the density unity, which, while simple, sufficiently illustrates the proposed mode of treating the subject.

We easily see that in the above simple case, $\alpha$ being any constant vector whatever, and $a$ being the radius of the sphere,

$$\iint ds\, \epsilon^{S\alpha\rho} = 2\pi a \int_{-a}^{+a} \epsilon^{xT\alpha} dx = \frac{2\pi a}{T\alpha} \left(\epsilon^{aT\alpha} - \epsilon^{-aT\alpha}\right).$$

Now, it appears that *we are at liberty to treat* $\Delta$ *as* $\alpha$ *has just been treated.* It is necessary, therefore, to find the effects of such operators as $T\Delta$, $\epsilon^{aT\Delta}$, &c., which seem to be novel, upon a scalar function of $T\sigma$; or $\mathfrak{T}$, as we may for the present call it.

Now $$(T\Delta)^2 F = -\Delta^2 F = F'' + \frac{2F'}{\mathfrak{T}},$$

whence it is easy to guess at a particular form of $T\Delta$. To make sure that it is the only one, assume

$$T\Delta = \mathrm{f}\frac{d}{d\mathfrak{T}} + \mathbf{f},$$

where $\mathrm{f}$ and $\mathbf{f}$ are scalar functions of $\mathfrak{T}$ to be found. This gives

$$(T\Delta)^2 F = \left(\mathrm{f}\frac{d}{d\mathfrak{T}} + \mathbf{f}\right)(\mathrm{f}F' + \mathbf{f}F)$$

$$= \mathrm{f}^2 F'' + (\mathrm{f}\mathrm{f}' + \mathbf{f}\mathrm{f} + \mathrm{f}\mathbf{f})F' + (\mathrm{f}\mathbf{f}' + \mathbf{f}^2)F.$$

Comparing, we have

$$\mathrm{f}^2 = 1,$$

$$\mathrm{f}\mathrm{f}' + \mathbf{f}\mathrm{f} + \mathrm{f}\mathbf{f} = \frac{2}{\mathfrak{T}};$$

$$\mathrm{f}\mathbf{f}' + \mathbf{f}^2 = 0.$$

From the first, $\mathrm{f} = \pm 1$,

whence the second gives $\mathbf{f} = \pm\dfrac{1}{\mathfrak{T}}$;

the signs of $\mathrm{f}$ and $\mathbf{f}$ being alike. The third is satisfied identically.

That is, finally, $\pm T\Delta = \dfrac{d}{d\mathfrak{T}} + \dfrac{1}{\mathfrak{T}}$.

Also, an easy induction shews that

$$\pm (T\Delta)^n = \left(\frac{d}{d\mathfrak{T}}\right)^n + \frac{n}{\mathfrak{T}}\left(\frac{d}{d\mathfrak{T}}\right)^{n-1}.$$

Hence we have at once

$$\epsilon^{aT\Delta} = 1 \pm a\left(\frac{d}{d\mathfrak{T}} + \frac{1}{\mathfrak{T}}\right) + \ldots \pm \frac{a^n}{1.2\ldots n}\left[\left(\frac{d}{d\mathfrak{T}}\right)^n + \frac{n}{\mathfrak{T}}\left(\frac{d}{d\mathfrak{T}}\right)^{n-1}\right] + \&c.$$

$$= \epsilon^{\pm a\frac{d}{d\mathfrak{T}}} \pm \frac{a}{\mathfrak{T}}\epsilon^{\pm a\frac{d}{d\mathfrak{T}}},$$

by the help of which we easily arrive at the well-known results. This we leave to the student*.

### O. *Applications of ∇ in connection with Calculus of Variations.*

518. We conclude with a few elementary examples of the use of ∇ in connection with the Calculus of Variations. These depend, for the most part, on the simple relation

$$\delta Q = -S\delta\rho\nabla . Q.$$

Let us first consider the expression

$$A = \int QTd\rho,$$

where $Td\rho$ is an element of a finite arc along which the integration extends, and the quaternion $Q$ is a function of $\rho$, generally a scalar.

---

* *Proc. R. S. E.*, 1871-2.

To shew the nature of the enquiry, note that if $Q$ be the speed of a unit particle, $A$ will be what is called the *Action*. If $Q$ be the potential energy per unit length of a chain, $A$ is the total potential energy. Such quantities are known to assume minimum, or at least "stationary", values in various physical processes.

We have for the variation of the above quantity

$$\begin{aligned}\delta A &= \int (\delta Q T d\rho + Q \delta T d\rho)\\ &= \int (\delta Q T d\rho - QS . U d\rho d\delta\rho)\\ &= -[QSU d\rho \delta\rho] + \int \{\delta Q T d\rho + S . \delta\rho d(QU d\rho)\},\end{aligned}$$

where the portion in square brackets refers to the limits only, and gives the terminal conditions. The remaining portion may easily be put in the form

$$S \int \delta\rho \{d(QU d\rho) - \nabla Q . T d\rho\}.$$

If the curve is to be determined by the condition that the variation of $A$ shall vanish, we must have, as $\delta\rho$ may have any direction,

$$d(QU d\rho) - \nabla Q . T d\rho = 0,$$

or, with the notation of Chap. X,

$$\frac{d}{ds}(Q\rho') - \nabla Q = 0.$$

This simple equation shews that (when $Q$ is a scalar)

(1) The osculating plane of the sought curve contains the vector $\nabla Q$.

(2) The curvature at any point is inversely as $Q$, and directly as the component of $\nabla Q$ parallel to the radius of absolute curvature.

519. As a first application, suppose $A$ to represent the *Action* of a unit particle moving freely under a system of forces which have a potential; so that

$$Q = T\dot{\rho},$$

and
$$\dot{\rho}^2 = 2(P - H),$$

where $P$ is the potential, $H$ the energy-constant.

These give
$$T\dot{\rho}\nabla T\dot{\rho} = Q\nabla Q = -\nabla P,$$

and
$$Q\rho' = \dot{\rho},$$

so that the equation above becomes simply

$$\ddot{\rho} + \nabla P = 0,$$

which is obviously the ordinary equation of motion of a free particle.

520. If we look to the superior limit only, the first expression for $\delta A$ becomes in the present case

$$-[T\dot{\rho}SUd\rho\delta\rho] = -S\dot{\rho}\delta\rho.$$

If we suppose a variation of the constant $H$, we get the following term from the unintegrated part

$$t\delta H.$$

Hence we have at once Hamilton's equations of *Varying Action* in the forms

$$\nabla A = \dot{\rho}$$

and $$\frac{dA}{dH} = t.$$

The first of these gives, by the help of the condition above,

$$(\nabla A)^2 = 2(P - H),$$

the well-known partial differential equation of the first order and second degree.

521. To shew that, if $A$ be any solution whatever of this equation, the vector $\nabla A$ represents the velocity in a free path capable of being described under the action of the given system of forces, we have

$$\begin{aligned}\frac{d}{dt}\dot{\rho} = \ddot{\rho} &= -\nabla P = -\tfrac{1}{2}\nabla(\nabla A)^2\\ &= -S(\nabla A.\nabla)\nabla A.\end{aligned}$$

But $$\frac{d}{dt}.\nabla A = -S(\dot{\rho}\nabla)\nabla A.$$

A comparison shews at once that the equality

$$\nabla A = \dot{\rho}$$

is consistent with each of these vector equations.

522. Again, if $\partial$ refer to the constants only,

$$\tfrac{1}{2}\partial(\nabla A)^2 = S.\nabla A\partial\nabla A = -\partial H$$

by the differential equation.

But we have also $$\frac{\partial A}{\partial H} = t,$$

which gives $$\frac{d}{dt}(\partial A) = -S(\dot{\rho}\nabla)\partial A = \partial H.$$

These two expressions for $\partial H$ again agree in giving

$$\nabla A = \dot{\rho},$$

and thus shew that the differential coefficients of $A$ with regard to the two constants of integration must, themselves, be constants. We thus have the equations of two surfaces whose intersection determines the path.

523. Let us suppose next that $A$ represents the *Time* of passage, so that the brachistochrone is required. Here we have

$$Q = \frac{1}{T\dot{\rho}},$$

the other condition being as in § 519, and we have

$$\frac{d}{dt}\dot{\rho}^{-1} - \dot{\rho}^{-2}\nabla P = 0,$$

which may be reduced to the symmetrical form

$$\ddot{\rho} + \dot{\rho}^{-1}\nabla P\dot{\rho} = 0.$$

It is very instructive to compare this equation with that of the free path as above, § 519; noting how the force $\nabla P$ is, as it were, *reflected* on the tangent of the path (§ 105). This is the well-known characteristic of such brachistochrones.

The application of Hamilton's method may be easily made, as in the preceding example. (Tait, *Trans. R. S. E.*, 1865.)

524. As a particular case, let us suppose gravity to be the only force, then

$$\nabla P = \alpha,$$

a constant vector, so that

$$\frac{d}{dt}\dot{\rho}^{-1} - \dot{\rho}^{-2}\alpha = 0.$$

The form of this equation suggests the assumption

$$\dot{\rho}^{-1} = \beta - p\alpha \tan qt,$$

where $p$ and $q$ are scalar constants, and

$$S\alpha\beta = 0.$$

Substituting, we get

$$-pq \sec^2 qt + (-\beta^2 - p^2\alpha^2 \tan^2 qt) = 0,$$

which gives $$pq = T^2\beta = p^2T^2\alpha.$$

Now let
$$p\beta^{-1}\alpha = \gamma;$$
this must be a unit-vector perpendicular to $\alpha$ and $\beta$, so that
$$\dot{\rho}^{-1} = \frac{\beta}{\cos qt}(\cos qt - \gamma \sin qt),$$
whence
$$\dot{\rho} = \cos qt\,(\cos qt + \gamma \sin qt)\,\beta^{-1}$$
(which may be verified at once by multiplication).

Finally, taking the origin so that the constant of integration may vanish, we have
$$2\rho\beta = t + \frac{1}{2q}(\sin 2qt - \gamma \cos 2qt),$$
which is obviously the equation of a cycloid referred to its vertex. The tangent at the vertex is parallel to $\beta$, and the axis of symmetry to $\alpha$. The equation, it should be noted, gives the law of description of the path.

525. In the case of a chain hanging under the action of given forces we have, as the quantity whose line-integral is to be a minimum,
$$Q = Pr,$$
where $P$ is the potential, $r$ the mass per unit-length.

Here we have also, of course,
$$\int T d\rho = l,$$
the length of the chain being given.

It is easy to see that this leads, by the method above, to the equation
$$\frac{d}{ds}\{(Pr + u)\,\rho'\} - r\nabla P = 0,$$
where $u$ is a scalar multiplier.

526. As a simple case, suppose the chain to be uniform. Then we may put $ru$ for $u$, and divide by $r$. Suppose farther that gravity is the only force, then
$$P = S\alpha\rho, \qquad \nabla P = -\alpha,$$
and
$$\frac{d}{ds}\{(S\alpha\rho + u)\,\rho'\} + \alpha = 0.$$

Differentiating, and operating by $S\rho'$, we find

$$S.\rho'\left\{\rho'\left(S\alpha\rho'+\frac{du}{ds}\right)+\alpha\right\}=0,$$

or, since $\rho'$ is a unit-vector,

$$\frac{du}{ds}=0,$$

which shews that $u$ is constant, and may therefore be allowed for by change of origin.

The curve lies obviously in a plane parallel to $\alpha$, and its equation is

$$(S\alpha\rho)^2+\alpha^2 s^2=\text{const.},$$

which is a well-known form of the equation of the common catenary.

When the quantity $Q$ of § 518 is a vector or a quaternion, we have simply an equation (like that there given) for *each* of the constituents.

527. Suppose $P$ and the constituents of $\sigma$ to be functions which vanish at the bounding surface of a simply-connected space $\Sigma$, or such at least that either $P$ or the constituents vanish there, the others (or other) not becoming infinite.

Then, by § 484,

$$\iiint d\varsigma S.\nabla(P\sigma)=\iint ds\,PS\sigma U\nu=0,$$

if the integrals be taken through and over $\Sigma$.

Thus $$\iiint d\varsigma\, S.\sigma\nabla P=-\iiint d\varsigma\, PS\nabla\sigma.$$

By the help of this expression we may easily prove a very remarkable proposition of Thomson (*Cam. and Dub. Math. Journal*, Jan. 1848, or *Reprint of Papers on Electrostatics*, § 206).

*To shew that there is one, and but one, solution of the equation*

$$S\nabla(e^2\nabla u)=4\pi r$$

*where* r *vanishes at an infinite distance, and* e *is any real scalar whatever, continuous or discontinuous.*

Let $v$ be the potential of a distribution of density $r$, so that

$$\nabla^2 v=4\pi r,$$

and consider the integral

$$Q=-\iiint d\varsigma\left(e\nabla u-\frac{1}{e}\nabla v\right)^2.$$

That $Q$ may be a minimum as depending on the value of $u$ (which is obviously possible since it cannot be negative, and since it may have any positive value, however large, if only greater than this minimum) we must have

$$0 = \tfrac{1}{2}\delta Q = -\iiint d\varsigma S.(e^2\nabla u - \nabla v)\nabla\delta u$$

By the lemma given above this may be written

$$= \iiint d\varsigma\, \delta u\, S\nabla\,(e^2\nabla u - \nabla v),$$

$$= \iiint d\varsigma\, \delta u\, \{S\nabla\,(e^2\nabla u) - 4\pi r\}.$$

Thus any value of $u$ which satisfies the given equation is such as to make $Q$ a minimum.

But there is only one value of $u$ which makes $Q$ a minimum; for, let $Q_1$ be the value of $Q$ when

$$u_1 = u + w$$

is substituted for this value of $u$, and we have

$$Q_1 = -\iiint d\varsigma\left(e\nabla\,(u+w) - \frac{1}{e}\nabla v\right)^2$$

$$= Q - 2\iiint d\varsigma\, S.(e^2\nabla u - \nabla v)\nabla w - \iiint d\varsigma\, e^2(\nabla w)^2.$$

The middle term of this expression may, by the proposition at the beginning of this section, be written

$$2\iiint d\varsigma\, w\,\{S\nabla(e^2\nabla u) - 4\pi r\},$$

and therefore vanishes. The last term is essentially positive. Thus if $u_1$ anywhere differ from $u$ (except, of course, by a constant quantity) it cannot make $Q$ a minimum; and therefore $u$ is a unique solution.

---

## MISCELLANEOUS EXAMPLES.

1. The expression

$$V\alpha\beta V\gamma\delta + V\alpha\gamma V\delta\beta + V\alpha\delta V\beta\gamma$$

denotes a vector. What vector?

2. If two surfaces intersect along a common line of curvature, they meet at a constant angle.

3. By the help of the quaternion formulae of rotation, translate into a new form the solution (given in § 248) of the problem of inscribing in a sphere a closed polygon the directions of whose sides are given.

4. Find the point, the sum of whose distances from any number of given points is a minimum.

If $\rho$ be the sought point, $\alpha_1$, $\alpha_2$, &c. the given points: shew that

$$\Sigma . U(\rho-\alpha)=0.$$

Give a dynamical illustration of this solution.

(*Proc. R. S. E.* 1866–7.)

5. Shew that

$$S . V\alpha\beta\gamma V\beta\gamma\alpha V\gamma\alpha\beta = 4S\alpha\beta\gamma S\alpha\beta S\beta\gamma S\gamma\alpha.$$

6. Express, in terms of the masses, and geocentric vectors of the sun and moon, the sun's vector disturbing force on the moon, and expand it to terms of the second order; pointing out the magnitudes and directions of the separate components.

(Hamilton, *Lectures*, p. 615.)

7. If $q=r^{\frac{1}{2}}$, shew that

$$\begin{aligned}
2\,dq &= 2dr^{\frac{1}{2}} = \tfrac{1}{2}(dr + Kq\,dr\,q^{-1})\,Sq^{-1} = \tfrac{1}{2}(dr+q^{-1}dr\,Kq)\,Sq^{-1}\\
&= (drq+Kqdr)\,q^{-1}(q+Kq)^{-1} = (drq+Kqdr)\,(r+Tr)^{-1}\\
&= \frac{dr+Uq^{-1}dr\,Uq^{-1}}{Tq\,(Uq+Uq^{-1})} = \frac{dr\,Uq+Uq^{-1}dr}{q\,(Uq+Uq^{-1})} = \frac{q^{-1}(Uqdr+dr\,Uq^{-1})}{Uq+Uq^{-1}}\\
&= \frac{q^{-1}(qdr+Trdrq^{-1})}{Tq\,(Uq+Uq^{-1})} = \frac{dr\,Uq+Uq^{-1}dr}{Tq\,(1+Ur)} = \frac{drKq^{-1}+q^{-1}dr}{1+Ur}\\
&= \left\{dr + V.Vdr\frac{V}{S}q\right\}q^{-1} = \left\{dr - V\left(Vdr\frac{V}{S}q^{-1}\right)\right\}q^{-1}\\
&= \frac{dr}{q} + V\left(V\frac{dr}{q}\frac{V}{S}q\right) = \frac{dr}{q} - V\left(V\frac{dr}{q}\frac{V}{S}q^{-1}\right)\\
&= drq^{-1} + V\,(Vq^{-1}.Vdr)\left(1+\frac{V}{S}q^{-1}\right):
\end{aligned}$$

and give geometrical interpretations of these varied expressions for the same quantity. (*Ibid.* p. 628.)

8. Shew that the equation of motion of a homogeneous solid of revolution about a point in its axis, which is not its centre of gravity, is

$$BV\rho\ddot{\rho} - A\Omega\dot{\rho} = V\rho\gamma,$$

where $\Omega$ is a constant. (*Trans. R. S. E.*, 1869.)

9. Find the point $P$, such that, if $A_1$, $A_2$, &c. be any fixed points in space, and $m_1$, $m_2$, &c. positive numerical quantities, $\Sigma . mAP$ shall be a minimum.

Shew that a closed (gauche) polygon can be constructed whose sides are parallel to $PA_1$, $PA_2$, &c. while their lengths are as $m_1$, $m_2$, &c., respectively.

If $T\Sigma . m\overline{AP}$ is to be a minimum, what is the result?

10. Form the quaternion condition that the lines joining the middle points of the sides of a closed polygon (plane or gauche) may form a similar polygon.

When this condition is satisfied, find the quaternion operator which must be applied to the second polygon to make it similarly situated with the first.

11. Solve the equations in linear and vector functions; $\varpi$ being given, while $\phi$ and $\chi$ are to be found:

$$(\alpha)\quad \phi^3 = \varpi,$$

$$(\beta)\quad \phi + \phi^2 = \varpi,$$

$$(\gamma)\quad \begin{cases} \chi + \phi = \varpi, \\ \phi\chi = \varpi^2. \end{cases}$$

12. Put the equation of a Minding line (§ 394) *directly* into the normal form for a line passing through each of two fixed curves:—

$$\rho = x\phi t + (1 - x)\psi u,$$

where $\phi$ and $\psi$ are definite vector functions of the arbitrary scalars $t$, and $u$, respectively.

13. Shew that

$$S\nabla\nabla_1 . \sigma\sigma_1 \quad \text{and} \quad \nabla\nabla_1 S\sigma\sigma_1$$

are different expressions for the same scalar, and give a number of other forms. (§ 513.) How are the values of these quantities respectively affected, if the $\nabla$ without suffix acts on $\sigma_1$ as well as on $\sigma$?

14. If (as in § 511)

$$S\psi\omega\psi\tau = S\omega\tau$$

for all values of $\omega$ and of $\tau$, shew by actual transformation, *not* by the obvious geometrical reasoning, that we have also

$$V\psi\omega\psi\tau = \psi V\omega\tau.$$

15. Shew that, to the third order in $T\beta$, the couple, due to any closed circuit, on a small magnet, $2\beta$, whose centre is at the origin, is proportional to

$$V.\beta\int\frac{V.\rho'}{T\rho^3}\left\{\rho\left(1-\frac{3}{2}\frac{T\beta^2}{T\rho^2}+\frac{15}{2}\frac{S^2\beta\rho}{T\rho^4}\right)+3\beta\frac{S\beta\rho}{T\rho^2}\right\}.$$

Simplify, by means of this formula, the quasi-Cartesian investigation of § 464.

16. Calculate the value of

$$\int\frac{V\rho\rho'}{T\rho^3}$$

for a circular current, the origin being taken at any point.

17. Find an approximate formula for the potential, near the centre of the field, when two equal, circular, currents have a common axis, and are at a distance equal to their common radius. (This is v. Helmholtz' Tangent Galvanometer.)

18. Deduce the various forms of Spherical Harmonics, and their relations, from the results of the application of scalar functions of $S\alpha\nabla$, and similar operators, to $1/T\rho$.

19. Integrate the differential equations:

$$(a)\quad \frac{dq}{dt}+aq=b,$$

$$(b)\quad \frac{d\rho}{dt}+\phi\rho=\alpha,$$

$$(c)\quad \frac{d^2\rho}{dt^2}+\phi\left(\frac{d\rho}{dt}\right)+\psi\rho=0;$$

where $a$ and $b$ are given quaternions, and $\phi$ and $\psi$ given linear and vector functions. (Tait, *Proc. R. S. E.*, 1870–1.)

20. Derive (4) of § 92 directly from (3) of § 91.

21. Find the successive values of the continued fraction

$$u_x=\left(\frac{j}{i+}\right)^x 0,$$

where $i$ and $j$ have their quaternion significations, and $x$ has the values 1, 2, 3, &c. (Hamilton, *Lectures*, p. 645.)

22. If we have

$$u_x = \left(\frac{j}{i+}\right)^x c,$$

where $c$ is a given quaternion, find the successive values.

For what values of $c$ does $u$ become constant? (*Ibid.* p. 652.)

23. Prove that the moment of hydrostatic pressures on the faces of any polyhedron is zero, ($a$) when the fluid pressure is the same throughout, ($b$) when it is due to any set of forces which have a potential.

24. What vector is given, in terms of two known vectors, by the relation

$$\rho^{-1} = \tfrac{1}{2}(\alpha^{-1} + \beta^{-1})\,?$$

Shew that the origin lies on the circle which passes through the extremities of these three vectors.

25. Tait, *Trans. and Proc. R. S. E.*, 1870–3.

With the notation of §§ 484, 495, prove

($a$) $\iiint S(\alpha\nabla)\,\tau\, d\varsigma = \iint \tau S\alpha U\nu ds.$

($b$) If $S(\rho\nabla)\,\tau = -\,n\tau,$

$$(n+3)\iiint \tau d\varsigma = -\iint \tau S\rho U\nu ds.$$

($c$) With the additional restriction $\nabla^2\tau = 0,$

$$\iint S\,.\,U\nu\,\{2n\rho + (n+3)\,\rho^2\nabla\}\,.\,\tau ds = 0.$$

($d$) Express the value of the last integral over a non-closed surface by a line-integral.

($e$) $-\int Td\rho = \iint ds S\,.\,U\nu\nabla\sigma,$

if $\sigma = Ud\rho$ all round the curve.

($f$) For any portion of surface whose bounding edge lies wholly on a sphere with the origin as centre

$$\iint ds S\,.\,(U\rho U\nu\nabla)\,.\,\sigma = 0,$$

whatever be the vector $\sigma$.

($g$) $\int Vd\rho\nabla\,.\,\sigma = \iint ds\,\{U\nu\nabla^2 - S(U\nu\nabla)\nabla\}\,\sigma,$

whatever be $\sigma$.

26. Tait, *Trans. R. S. E.*, 1873.

Interpret the equation

$$d\sigma = uqd\rho q^{-1},$$

and shew that it leads to the following results

$$\nabla^2\sigma = q\nabla uq^{-1},$$

$$\nabla . uq^{-1} = 0,$$

$$\nabla^2 . u^{\frac{1}{2}} = 0.$$

Hence shew that the only sets of surfaces which, together, cut space into cubes are planes and their electric images.

27. What problem has its conditions stated in the following six equations, from which $\xi$, $\eta$, $\zeta$ are to be determined as scalar functions of $x$, $y$, $z$, or of

$$\rho = ix + jy + kz\,?$$

$$\nabla^2\xi = 0, \qquad \nabla^2\eta = 0, \qquad \nabla^2\zeta = 0,$$

$$S\nabla\xi\nabla\eta = 0, \qquad S\nabla\eta\nabla\zeta = 0, \qquad S\nabla\zeta\nabla\xi = 0,$$

where
$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

Shew that they give the farther equations

$$0 = \nabla^2\xi\eta = \nabla^2\eta\zeta = \nabla^2\zeta\xi = \nabla^2 . \xi\eta\zeta.$$

Shew that (with a change of origin) the general solution of these equations may be put in the form

$$S\rho\,(\phi + f)^{-1}\rho = 1,$$

where $\phi$ is a self-conjugate linear and vector function, and $\xi$, $\eta$, $\zeta$ are to be found respectively from the three values of $f$ at any point by relations similar to those in Ex. 24 to Chapter X.

28. Shew that, if $\rho$ be a planet's radius vector, the potential $P$ of masses external to the solar system introduces into the equation of motion a term of the form $S(\rho\nabla)\nabla P$.

Shew that this is a self-conjugate linear and vector function of $\rho$, and that it involves only *five* independent constants.

Supposing the undisturbed motion to be circular, find the chief effects which this disturbance can produce.

29. In § 430 above, we have the equations

$$V\alpha(\ddot{\varpi} + n^2\varpi) = 0, \quad S\alpha\varpi = 0, \quad \dot{\alpha} = \omega Vi\alpha, \quad T\alpha = 1,$$

where $\omega^2$ is neglected. Shew that with the assumptions

$$q = i^{\frac{\omega t}{\pi}}, \quad \alpha = q\beta q^{-1}, \quad r = \beta^{\frac{\omega_1 t}{\pi}}, \quad \varpi = qr\tau r^{-1}q^{-1},$$

we have

$$\dot{\beta} = 0, \quad T\beta = 1, \quad S\beta\tau = 0, \quad V\beta(\ddot{\tau} + n^2\tau) = 0,$$

provided $\omega Si\alpha - \omega_1 = 0$. Hence deduce the behaviour of the

Foucault pendulum without the $x$, $y$, and $\xi$, $\eta$ transformations in the text.

Apply analogous methods to the problems proposed at the end of § 426 of the text.

30. Hamilton, *Bishop Law's Premium Examination*, 1862.

(*a*) If OABP be four points of space, whereof the three first are given, and not collinear; if also

$$\text{OA} = \alpha, \ \text{OB} = \beta, \ \text{OP} = \rho;$$

and if, in the equation

$$F\frac{\rho}{\alpha} = F\frac{\beta}{\alpha},$$

the characteristic of operation $F$ be replaced by $S$, the locus of P is a plane. What plane?

(*b*) In the same general equation, if $F$ be replaced by $V$, the locus is an indefinite right line. What line?

(*c*) If $F$ be changed to $K$, the locus of P is a point. What point?

(*d*) If $F$ be made $= U$, the locus is an indefinite half-line, or ray. What ray?

(*e*) If $F$ be replaced by $T$, the locus is a sphere. What sphere?

(*f*) If $F$ be changed to $TV$, the locus is a cylinder of revolution. What cylinder?

(*g*) If $F$ be made $TVU$, the locus is a cone of revolution. What cone?

(*h*) If $SU$ be substituted for $F$, the locus is one sheet of such a cone. Of what cone? and which sheet?

(*i*) If $F$ be changed to $VU$, the locus is a pair of rays. Which pair?

31. Hamilton, *Bishop Law's Premium Examination*, 1863.

(*a*) The equation

$$S\rho\rho' + a^2 = 0$$

expresses that $\rho$ and $\rho'$ are the vectors of two points P and P′, which are conjugate with respect to the sphere

$$\rho^2 + a^2 = 0;$$

or of which one is on the polar plane of the other.

(*b*) Prove by quaternions that if the right line PP′, connecting two such points, intersect the sphere, it is cut harmonically thereby.

(*c*) If P′ be a given external point, the cone of tangents drawn from it is represented by the equation,

$$(V\rho\rho')^2 = a^2(\rho - \rho')^2;$$

and the orthogonal cone, concentric with the sphere, by

$$(S\rho\rho')^2 + a^2\rho^2 = 0.$$

(*d*) Prove and interpret the equation,

$$T(n\rho - \alpha) = T(\rho - n\alpha), \text{ if } T\rho = T\alpha.$$

(*e*) Transform and interpret the equation of the ellipsoid,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2.$$

(*f*) The equation

$$(\kappa^2 - \iota^2)^2 = (\iota^2 + \kappa^2)S\rho\rho' + 2S\iota\rho\kappa\rho'$$

expresses that $\rho$ and $\rho'$ are values of conjugate points, with respect to the same ellipsoid.

(*g*) The equation of the ellipsoid may also be thus written,

$$S\nu\rho = 1, \text{ if } (\kappa^2 - \iota^2)^2\nu = (\iota - \kappa)^2\rho + 2\iota S\kappa\rho + 2\kappa S\iota\rho.$$

(*h*) The last equation gives also,

$$(\kappa^2 - \iota^2)^2\nu = (\iota^2 + \kappa^2)\rho + 2V\iota\rho\kappa.$$

(*i*) With the same signification of $\nu$, the differential equations of the ellipsoid and its reciprocal become

$$S\nu d\rho = 0, \qquad S\rho d\nu = 0.$$

(*j*) Eliminate $\rho$ between the four scalar equations,

$$S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c, \quad S\epsilon\rho = e.$$

32. Hamilton, *Bishop Law's Premium Examination*, 1864.

(*a*) Let $A_1B_1$, $A_2B_2$, … $A_nB_n$ be any given system of posited right lines, the $2n$ points being all given; and let their vector sum,

$$AB = A_1B_1 + A_2B_2 + \ldots + A_nB_n,$$

be a line which does not vanish. Then a point $H$, and a scalar $h$, can be determined, which shall satisfy the quaternion equation,

$$HA_1 \,.\, A_1B_1 + \ldots + HA_n \,.\, A_nB_n = h \,.\, AB;$$

namely by assuming any origin $O$, and writing,

$$OH = V\frac{OA_1 \,.\, A_1B_1 + \ldots + OA_n \,.\, A_nB_n}{A_1B_1 + \ldots + A_nB_n},$$

$$h = S\frac{OA_1 \,.\, A_1B_1 + \ldots}{A_1B_1 + \ldots}.$$

(*b*) For any assumed point $C$, let

$$Q_C = CA_1 \,.\, A_1B_1 + \ldots + CA_n \,.\, A_nB_n ;$$

then this quaternion sum may be transformed as follows,

$$Q_C = Q_H + CH \,.\, AB = (h + CH) \,.\, AB ;$$

and therefore its tensor is

$$TQ_C = (h^2 + \overline{CH}^2)^{\frac{1}{2}} \,.\, \overline{AB},$$

in which $\overline{AB}$ and $\overline{CH}$ denote lengths.

(*c*) The least value of this tensor $TQ_C$ is obtained by placing the point $C$ at $H$; if then a quaternion be said to be a minimum when its tensor is such, we may write

$$\text{min. } Q_C = Q_H = h \,.\, AB ;$$

so that this minimum of $Q_C$ is a vector.

(*d*) The equation

$$TQ_C = c = \text{any scalar constant} > TQ_H$$

expresses that the locus of the variable point $C$ is a spheric surface, with its centre at the fixed point $H$, and with a radius $r$, or $\overline{CH}$, such that

$$r \,.\, \overline{AB} = (TQ_C^2 - TQ_H^2)^{\frac{1}{2}} = (c^2 - h^2 \,.\, \overline{AB}^2)^{\frac{1}{2}} ;$$

so that $H$, as being thus the common centre of a series of concentric spheres, determined by the given system of right lines, may be said to be the *Central Point*, or simply the *Centre*, of that system.

(*e*) The equation

$$TVQ_C = c_1 = \text{any scalar constant} > TQ_H$$

represents a right cylinder, of which the radius

$$= (c_1^{\,2} - h^2 \,.\, \overline{AB}^2)^{\frac{1}{2}}$$

divided by $\overline{AB}$, and of which the axis of revolution is the line,

$$VQ_C = Q_H = h \,.\, AB ;$$

wherefore this last right line, as being the common axis of a series of such right cylinders, may be called the *Central Axis* of the system.

(*f*) The equation

$$SQ_C = c_2 = \text{any scalar constant}$$

represents a plane; and all such planes are parallel to the *Central Plane*, of which the equation is

$$SQ_C = 0.$$

($g$) Prove that the central axis intersects the central plane perpendicularly, in the central point of the system.

($h$) When the $n$ given vectors $A_1B_1, \ldots A_nB_n$ are parallel, and are therefore proportional to $n$ scalars, $b_1, \ldots b_n$, the scalar $h$ and the vector $Q_H$ vanish; and the centre $H$ is then determined by the equation

$$b_1 . HA_1 + b_2 . HA_2 + \ldots + b_n . HA_n = 0,$$

or by the expression,

$$OH = \frac{b_1 . OA_1 + \ldots + b_n . OA_n}{b_1 + \ldots + b_n},$$

where $O$ is again an arbitrary origin.

33. Hamilton, *Bishop Law's Premium Examination*, 1860.

($a$) The normal at the end of the variable vector $\rho$, to the surface of revolution of the sixth dimension, which is represented by the equation

$$(\rho^2 - \alpha^2)^3 = 27\alpha^2 (\rho - \alpha)^4 \ldots\ldots\ldots\ldots\ldots\ldots\text{(a)},$$

or by the system of the two equations,

$$\rho^2 - \alpha^2 = 3t^2\alpha^2, \quad (\rho - \alpha)^2 = t^3\alpha^2 \ldots\ldots\ldots\ldots\ldots\text{(a}'\text{)},$$

and the tangent to the meridian at that point, are respectively parallel to the two vectors,

$$\nu = 2(\rho - \alpha) - t\rho,$$

and

$$\tau = 2(1 - 2t)(\rho - \alpha) + t^2\rho;$$

so that they intersect the axis $\alpha$, in points of which the vectors are, respectively,

$$\frac{2\alpha}{2 - t}, \text{ and } \frac{2(1 - 2t)\alpha}{(2 - t)^2 - 2}.$$

($b$) If $d\rho$ be in the same meridian plane as $\rho$, then

$$t(1 - t)(4 - t)\, d\rho = 3\tau dt, \text{ and } S\frac{\rho dt}{d\rho} = \frac{4 - t}{3}.$$

($c$) Under the same condition,

$$S\frac{d\nu}{d\rho} = \frac{2}{3}(1 - t).$$

($d$) The vector of the centre of curvature of the meridian, at the end of the vector $\rho$, is, therefore,

$$\sigma = \rho - \nu\left(S\frac{d\nu}{d\rho}\right)^{-1} = \rho - \frac{3}{2}\frac{\nu}{1 - t} = \frac{6\alpha - (4 - t)\rho}{2(1 - t)}.$$

(*e*) The expressions in (*a*) give

$$\nu^2 = \alpha^2 t^2 (1-t)^2, \quad \tau^2 = \alpha^2 t^3 (1-t)^2 (4-t);$$

hence $$(\sigma-\rho)^2 = \frac{9}{4}\alpha^2 t^2, \text{ and } d\rho^2 = \frac{9\alpha^2 t}{4-t} dt^2;$$

the radius of curvature of the meridian is, therefore,

$$R = T(\sigma-\rho) = \frac{3}{2} tT\alpha;$$

and the length of an element of arc of that curve is

$$ds = Td\rho = 3T\alpha \left(\frac{t}{4-t}\right)^{\frac{1}{2}} dt.$$

(*f*) The same expressions give

$$4(V\alpha\rho)^2 = -\alpha^4 t^3 (1-t)^2 (4-t);$$

thus the auxiliary scalar $t$ is confined between the limits 0 and 4, and we may write $t = 2 \text{ vers } \theta$, where $\theta$ is a real angle, which varies continuously from 0 to $2\pi$; the recent expression for the element of arc becomes, therefore,

$$ds = 3T\alpha \,.\, td\theta,$$

and gives by integration

$$s = 6T\alpha(\theta - \sin\theta),$$

if the arc $s$ be measured from the point, say $F$, for which $\rho = \alpha$, and which is common to all the meridians; and the total periphery of any one such curve is $= 12\pi T\alpha$.

(*g*) The value of $\sigma$ gives

$$4(\sigma^2-\alpha^2) = 3\alpha^2 t(4-t), \quad 16(V\alpha\sigma)^2 = -\alpha^4 t^3 (4-t)^3;$$

if, then, we set aside the axis of revolution $\alpha$, which is *crossed* by all the normals to the surface (a), the surface of centres of curvature which is *touched* by all those normals is represented by the equation,

$$4(\sigma^2-\alpha^2)^3 + 27\alpha^2 (V\alpha\sigma)^2 = 0 \,\ldots\ldots\ldots\ldots\ldots\ldots\text{(b)}.$$

(*h*) The point $F$ is common to the two surfaces (a) and (b), and is a singular point on each of them, being a triple point on (a), and a double point on (b); there is also at it an infinitely sharp cusp on (b), which tends to coincide with the axis $\alpha$, but a determined tangent plane to (a), which is perpendicular to that axis, and to that cusp; and the point, say $F'$, of which the vector $= -\alpha$, is another and an exactly similar cusp on (b), but does not belong to (a).

(*i*) Besides the *three* universally *coincident* intersections of the surface (a), with *any* transversal, drawn through its triple point $F$, in *any* given direction $\beta$, there are always *three other real intersections*, of which indeed one coincides with $F$ if the transversal be perpendicular to the axis, and for which the following is a general formula:

$$\rho = T\alpha . [U\alpha + \{2SU(\alpha\beta)^{\frac{1}{3}}\}^3 U\beta].$$

(*j*) The point, say $V$, of which the vector is $\rho = 2\alpha$, is a double point of (a), near which that surface has a cusp, which coincides nearly with its tangent cone at that point; and the semi-angle of this cone is $= \frac{\pi}{6}$.

*Auxiliary Equations:*

$$\begin{cases} 2S\rho(\rho - \alpha) = \alpha^2 t^2 (3 + t), \\ 2S\alpha(\rho - \alpha) = \alpha^2 t^2 (3 - t). \end{cases}$$

$$\begin{cases} S\nu\rho = -\alpha^2 t (1 - t)(1 - 2t), \\ 2S\nu(\rho - \alpha) = \alpha^2 t^3 (1 - t). \end{cases}$$

$$\begin{cases} S\rho\tau = \alpha^2 t^2 (1 - t)(4 - t), \\ 2S(\rho - \alpha)\tau = \alpha^2 t^3 (1 - t)(4 - t). \end{cases}$$

34. A homogeneous function of $\rho$, of the $n$th degree, is changed into the same function of the constant vector, $\alpha$, by the operator

$$\frac{1}{n!} T . (S\alpha\nabla)^n.$$

35. If $r$ be the cube root of the quaternion $q$, shew that

$$dr = p + (V . r^2 + rVr) Vq^{-1}(rp - pr),$$

where $p = \frac{1}{3} r^{-2} dq$. (Hamilton, *Lectures*, p. 629.)

36. Shew from § 509 that the term to be added to $\nabla Q$ of § 512 (3), in consequence of viscosity, is proportional to

$$\nabla^2 \sigma + \tfrac{1}{3} \nabla S \nabla \sigma.$$

37. Shew, from structural considerations, that

$$V . (S\alpha\nabla . \sigma S\beta\nabla_1 . \sigma_1)$$

must be a linear and vector function of $V\alpha\beta$. Also prove, directly, that its value is

$$-\tfrac{1}{2} S . (V\alpha\beta)\nabla\nabla_1 . V\sigma\sigma_1.$$

38. The spherical opening subtended at $\alpha$ by the sphere

$$T\rho = r$$

is
$$-\frac{1}{r}\iint ds\,\frac{S\rho(\rho-\alpha)}{T(\rho-\alpha)^3} = 4\pi, \text{ or } 0,$$

according as $T\alpha = a \lesseqgtr r$.

Hence shew, without further integration, that (with the same conditions)

$$\iint \frac{ds}{T(\rho-\alpha)} = 4\pi r, \text{ or } \frac{4\pi r^2}{a},$$

whence, of course,

$$\iint \frac{ds\,U(\rho-\alpha)}{T(\rho-\alpha)^2} = 0 \text{ or } -\frac{4\pi r^2 U\alpha}{a^2}.$$

Also that

$$\iint \frac{ds}{T(\rho-\alpha)^3} = \frac{4\pi r}{r^2-a^2}, \text{ or } \frac{4\pi r^2}{a(a^2-r^2)}.$$

39. Find also the values of

$$\iint \frac{U\nu ds}{T(\rho-\alpha)}, \iint \frac{U\nu ds}{T(\rho-\alpha)^3}, \text{ and such-like,}$$

the integration extending over the surface of the sphere $T\rho = r$.

40. Shew that the potential at $\beta$, due to mass $m$ at $\alpha$, is to the potential at $\beta^{-1}$, due to mass $m'$ at $\alpha^{-1}$, as $1 : T\beta$, provided $m' : m :: 1 : T\alpha$.

Hence, by the results of 38 above, shew that, with $a < r$,

$$\iint \frac{ds}{T(\rho-\beta)\,T(\rho-\alpha)^3} = \frac{4\pi r}{(r^2-a^2)\,T(\beta-\alpha)}, \text{ or } \frac{4\pi r^2}{a(r^2-a^2)\,T(r^2\alpha^{-1}+\beta)}$$

according as $T\beta \gtrless r$, the limits of integration being as before.

Obtain these results directly from Green's Theorem (§ 486) without employing the Electric Image transformation.

41. Shew that the quaternion

$$\iint \sigma S(U\nu\nabla)\,\tau ds - \iiint \sigma\nabla^2\tau d\varsigma$$

(in the notation of § 482) is changed into its own conjugate by interchange of $\sigma$ and $\tau$. Express its value (by §§ 133, 486) as a single volume-integral.

42. Prove that

$$\iiint(\sigma\nabla^2\tau + K\nabla\sigma\,.\,\nabla\tau)\,d\varsigma = \iint\sigma U\nu\nabla\tau ds,$$

and thence that

$$\iiint(\tau\nabla^2\sigma - \nabla^2\tau\,.\,\sigma)\,d\varsigma = \iint(\tau U\nu\nabla\sigma - K\nabla\tau\,.\,U\nu\sigma)\,ds.$$

43. Find the distribution of matter on a given closed surface which will produce, in its interior, the same potential as does a given distribution of matter outside it.

Hence shew that there is one, and only one, distribution of matter over a surface, which will produce, at each point of it, any arbitrarily assigned potential.

---

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AND SONS, AT THE UNIVERSITY PRESS.

## *By the same Author.*

**A Treatise on Natural Philosophy.** By Sir W. Thomson, LL.D., D.C.L., F.R.S., Professor of Natural Philosophy in the University of Glasgow, and P. G. Tait, M.A., Professor of Natural Philosophy in the University of Edinburgh. **Part I.** Demy 8vo. 16*s*. **Part II.** Demy 8vo. 18*s*.

**Elements of Natural Philosophy.** By Professors Sir W. Thomson and P. G. Tait. Demy 8vo. 9*s*.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

**Lectures on some Recent Advances in Physical Science.** With Illustrations. Revised and enlarged, with the Lecture on Force delivered before the British Association. By P. G. Tait, M.A. Third Edition. Crown 8vo. 9*s*.

**Heat.** With numerous Illustrations. By P. G. Tait, M.A. Crown 8vo. 6*s*.

**A Treatise on Dynamics of a Particle.** With numerous Examples. By P. G. Tait, M.A., and the late W. J. Steele, B.A., Fellow of St Peter's College, Cambridge. Sixth Edition, carefully Revised. Crown 8vo. 12*s*.

**The Unseen Universe; or, Physical Speculations on a Future State.** By Professors Balfour Stewart and P. G. Tait. Fifteenth Edition. Crown 8vo. 6*s*.

MACMILLAN AND CO. LONDON.

**Light.** By P. G. Tait, M.A., Sec. R.S.E. Second Edition. Revised and Enlarged. Crown 8vo. 6*s*.

**Properties of Matter.** By P. G. Tait, M.A., Sec. R.S.E. Crown 8vo. 7*s*. 6*d*.

ADAM & CHARLES BLACK, EDINBURGH.

QA 196.T3 1890 sci
[redacted]

P8-CTQ-469